TOPOLOGICALLY TWISTED SUPERSYMMETRIC GAUGE THEORIES:

INVARIANTS OF 3–MANIFOLDS, QUANTUM INTEGRABLE SYSTEM, THE 3D/3D CORRESPONDENCE AND BEYOND

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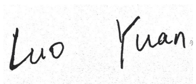
Declaration

I hereby declare that the thesis is based on original work done by myself (jointly with others). I have duly acknowledged all the sources of information which have been used in the thesis.

This thesis has also not been submitted for any degree in any university previously.



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Abstract

We construct and explore a variety of topologically twisted supersymmetric gauge theories, which result in various inspiring applications in both physics and mathematics, ranging within the following three cases.

In the first case, we construct a topological Chern-Simons sigma model on a Riemannian three-manifold M with gauge group $\mathcal{G}$ whose hyperkähler target space X is equipped with a $\mathcal{G}$-action. Via a perturbative computation of its partition function, we obtain topological invariants of M that define new weight systems which are characterized by both Lie algebra structure *and* hyperkähler geometry. In canonically quantizing the sigma model, we find that the partition function on certain M can be expressed in terms of Chern-Simons knot invariants of M and the intersection number of certain $\mathcal{G}$-equivariant cycles in the moduli space of $\mathcal{G}$-covariant maps from M to X. We also construct supersymmetric Wilson loop operators, and via a perturbative computation of their expectation value, we obtain knot invariants of M that define new knot weight systems which are also characterized by both Lie algebra structure *and* hyperkähler geometry.

In the second case, we study an $\mathcal{N} = 2$ supersymmetric gauge theory on the product of a two-sphere and a cylinder, which is topologically twisted along the cylinder. By localization on the two-sphere, we show that the low-energy dynamics of a BPS sector of such a theory is described by a quantum integrable system, with the Planck constant set by the inverse of the radius of the sphere. If the sphere is replaced with a hemisphere, then our system reduces to an integrable system of the type studied by Nekrasov and Shatashvili. In this case we establish a correspondence between the effective prepotential of the gauge theory and the Yang-Yang function of the integrable system.

In the last case, we formulate a five-dimensional super-Yang-Mills theory (SYM) on $\mathbb{D}^2 \times M$, which has a single supercharge Q, and Q is topologically twisted along the three-manifold M and is the Ω-deformation of the B-twisted $N = (2, 2)$ supercharges on the disk $\mathbb{D}^2$. Our 5d SYM can be viewed as the compactification of the 6d $(2, 0)$ superconformal field theory on S^1. By localization on $\mathbb{D}^2$, our 5d SYM reduces to the holomorphic part of the complex

Chern-Simons theory. As a consequence, our result indicates the existence of a mirror symmetry in two-dimensional Ω-deformed gauge theories.

This thesis is based on the work reported in the following papers:

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Y. Luo, M.-C. Tan, and J. Yagi, $\mathcal{N} = 2$ *supersymmetric gauge theories and quantum integrable systems*, *JHEP* **1403** (2014) 090 [arXiv:1310.0827].

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Contents

Chapter 1

Introduction

Supersymmetric quantum field theories, despite the strong constraints imposed by their supersymmetries, are usually not exactly solvable due to various quantum corrections. However, if we compute the theories constrained in certain BPS sectors, which preserve the corresponding supercharges that are usually topologically twisted, the exact solutions can be found with affordable efforts. The topological twisting turns a certain supercharge Q into a scalar on the spacetime manifold; and with respect to Q, one can construct a topologically twisted theory that corresponds to a certain BPS sector of the untwisted theory. To evaluate these theories, one can use localization techniques to perform path-integral computations, whereby the field configurations localize to vacuum configurations and the quantum corrections only need to be considered up to the one-loop order in perturbation theory. Thus, the partition function and Q-invariant correlation functions can be computed exactly. Such an advantage makes topologically twisted theories very powerful models in both physics and mathematics research. Within the wide range of their applications, this thesis mainly focuses on the following three topics.

First, since the field configurations are localized to the vacua, these theories are good candidates for studying low-energy physics and can reveal many intriguing properties of low-energy physics. Second, as the BPS sector which preserves the scalar supercharge is protected against dimensional reductions, two different theories in lower dimensions that are reduced from a topologically twisted theory in higher dimensions are equivalent to each other under identification of

Q-invariant quantities, revealing various correspondences in physics. Third, besides their inspiring applications in physics, topologically twisted theories build a solid bridge between physics and mathematics. Since their invention in the late 1980s [1, 2], topologically twisted theories have borne rich fruit in mathematics, mostly in topology. The results of this thesis lie within the range of these three areas, and as we shall see, our results enrich them in varied aspects.

In summary, we formulate and explore a variety of supersymmetric gauge theories, where the theories are topologically twisted or partially twisted along certain manifolds. In studying these theories via localization or some nonperturbative methods, we construct new topological invariants of 3-manifolds, obtain quantum integrable systems, and gain a deeper understanding of a correspondence between two three-dimensional theories. A brief introduction of these three cases is given in the following.

Three-Manifold Invariants from 3d Chern-Simons Sigma Model

In this case we focus on the topic of relating physics to mathematics. We construct a Chern-Simons sigma model in three dimensions. This model is a topological quantum field theory (TQFT) with a scalar supercharge.

For the topological field theory, on the physical side, the correlation functions of the Q-invariant operators are metric-independent. So in term of mathematics, as they are independent of the metric variations, these correlation functions are topological invariants. Therefore, the TQFT setup provides a powerful toolbox for constructing and studying the topological invariants, on the mathematical side. To elaborate on this point, let us have a brief review of the history of TQFTs.

The seminal work on TQFTs was done by E. Witten [1] in 1988. By topological twisting the $\mathcal{N} = 2$ super-Yang-Mills theory, he constructed the topological theory now known as Donaldson-Witten theory. Witten showed that its Q-invariant correlation functions are actually the Donaldson invariants of four manifolds. Around the same time, Witten also formulated another two different TQFTs: the two-dimensional topological sigma model [2] and three-dimensional

Chern-Simons gauge theory [3]. Witten found that these two theories can be applied to study a variety of topological invariants: Gromov invariants [4], as well as knot and link invariants (the Jones polynomial [5] and its generalizations).

These various topological field theories can be divided into two categories: Schwarz type (whose action is metric-independent per se) and Witten type (whose action is metric-dependent but in a Q-exact form, with topologically twisted supercharge Q). Among theories of the Schwarz type, three-dimensional Chern-Simons theory is one of the most celebrated. Following the path opened up by Witten [3], further developments [6–9] deepened the study of topological invariants of three-manifolds via Chern-Simons theory: weight systems whose weights depend on the Lie algebra structure underlying the gauge group were constructed to express certain three-manifold invariants. Inspired by these developments, Rozansky and Witten sought, and successfully found a weight system whose weights depend on hyperkähler geometry instead of Lie algebra structure, by computing the partition function of a certain three-dimensional supersymmetric topological sigma model with a hyperkähler target space [10], which is a Witten type TQFT.

Encouraged by the success of the two theories, people sought to construct more exotic three-manifold invariants that can be expressed as weight systems whose weights depend on both Lie algebra structure and hyperkähler geometry, by studying, naturally, the hybrids of Chern-Simons theory and the Rozansky-Witten sigma model – the topological Chern-Simons sigma models [11–13]. This is also the direction that we take in chapter 2. We construct an appropriate topological Chern-Simons sigma model, studying which, we formulate and discuss novel three-manifold invariants, their knot generalizations, and beyond.

Low Energy Effective Theories and Integrable Systems

In another more physical perspective, constraining BPS sectors within certain topological sectors, topological twisting can be used to study low energy dynamics of supersymmetric field theories.

Contrary to the difficulties of exactly solving untwisted supersymmetric theories, a nice feature of topologically twisted theories is the existence of exact solutions, as the topological twisting keeps only the low energy information of the theories. Thus, topological twisting gives us a powerful tool for obtaining effective theories in the low energy limit and studying low energy physics. And importantly, many physically interesting questions are related to the vacuum structure of the untwisted theories and therefore can be answered by studying the low energy effective theories.

Among the effective theories of supersymmetric gauge theories, Seiberg-Witten theory [14] is one of the best known examples. Seiberg and Witten constructed the low energy effective theory for four-dimensional $\mathcal{N} = 2$ supersymmetric gauge theories with gauge group $SU(2)$. They exactly described the moduli space of the vacua of the theories. Not long after this seminal work, it was realized that there exists a connection between Seiberg-Witten theories and complex integrable systems [15–22]. A few years later, Nekrasov and Shatashvili [23] found that turning on a certain deformation (which is called the Ω-deformation [24]) on a two-plane quantizes these integrable systems, with the deformation parameter ε playing the role of the Planck constant. An explanation of this result was subsequently given by Nekrasov and Witten [25] using a brane construction.

In chapter 3, we establish another, yet closely related, connection between $\mathcal{N} = 2$ supersymmetric gauge theories and quantum integrable systems. Instead of turning on Ω-deformation, we compactify a two-plane to a round two-sphere S^2 of radius r. One of the remaining two dimensions is compactified to a circle S^1; therefore our setup is an $\mathcal{N} = 2$ supersymmetric gauge theory formulated on $S^2 \times \mathbb{R} \times S^1$. We find that the low-energy dynamics of a BPS sector of this theory is described by a quantum integrable system, with the Planck constant set by $1/r$. This system quantizes the real integrable system whose symplectic form is $\mathrm{Re}(\Omega)$, where Ω is the holomorphic symplectic form of the complex integrable system associated to the Coulomb branch.

Deciphering 3d/3d Correspondence via 5d Super-Yang-Mills

The last topic of this thesis also has to do with the fact that the topologically twisted theories consider only the Q-invariant sectors of untwisted theories, with topologically twisted supercharges. Since the Q-invariant quantities can be preserved under dimensional reduction, we can apply such theories to resolve some intriguing correspondences in physics, as elaborated in the following.

In 2009, Alday, Gaiotto and Tachikawa [26] discovered a correspondence between $\mathcal{N}=2$ superconformal gauge theory in four dimensions and Liouville theory in two dimensions, which has been known as the AGT correspondence and studied extensively [27–30] since then. A few years later, a related correspondence between three-dimensional theories has been found [31–34], whereby two classes of quantum field theories are related: 3d $\mathcal{N}=2$ superconformal field theories (SCFTs) and 3d Chern-Simons theories with complex gauge group. From a wider perspective, such 4d/2d and 3d/3d correspondences both belong to the set of various correspondences between supersymmetric theories in d dimensions and nonsupersymmetric theories in $6-d$ dimensions. And it is widely believed that these $d/(6-d)$ correspondences have a common origin from $\mathcal{N}=(2,0)$ SCFTs in six dimensions. For the 4d/2d correspondence, considering a 6d $\mathcal{N}=(2,0)$ SCFT on $S^4\times M$, with M a punctured Riemann surface, the 4d and 2d theories in the AGT correspondence can be obtained respectively via compactification on M and localization on S^4 of the 6d theory [35–39]. The correspondence can be established by identifying the quantities preserved under these two procedures.

As for the 3d/3d correspondence, despite the complexity of performing explicit compactification on a general three-manifold, deriving the complex Chern-Simons theory by the localization has been more or less achieved by various works [40–43]. Our paper is also dedicated to trying to decipher the 3d/3d correspondence from the 6d viewpoint, using a typical yet fresh setup, where the novelty of our construction is that we equip the spacetime with an Ω-background. We place the theory on $(S^1\times_\varepsilon \mathbb{D}^2)\times M$, where $\mathbb{D}^2$ denotes a disk and ε is the Ω-deformation parameter. However, the 6d $N=(2,0)$ theory has no known Lagrangian, so we actually construct a super-Yang-Mills theory on $\mathbb{D}^2_\varepsilon\times M$ which is the dimensional reduction of the 6d theory on the S^1. By localization of the

5d SYM on $\mathbb{D}^2$ we obtain the holomorphic part of complex Chern-Simons theory on M. This will be the main theme of chapter 4 of this thesis.

Chapter 2

A Topological Chern-Simons Sigma Model and New Invariants of Three-Manifolds

2.1 Introduction

In this chapter, we will show that a 3d topological field theory results in intriguing applications in three-dimensional topology. We construct a topological supersymmetric Chern-Simons sigma model in three dimensions. Studying this model, we formulate and discuss novel invariants of three-manifolds. Let us first give a brief introduction on three-dimensional TQFTs and their applications in topology.

2.1.1 Background and Motivation

As mentioned in chapter 1, the relevance of three-dimensional quantum field theory – in particular, topological Chern-Simons gauge theory – to the study of three-manifold invariants, was first elucidated in a seminal paper by Witten [3] in an attempt to furnish a three-dimensional interpretation of the Jones polynomial [44] of knots in three-space. Along this direction, further developments [6–9] culminated in the observation that certain three-manifold invariants can be expressed as weight systems whose weights depend on the Lie algebra structure which underlies the gauge group. Since these weights are naturally associated to Feynman diagrams via their relation to Chern-Simons theory, it meant that such three-manifold invariants have an alternative interpretation as Lie algebra-dependent graphical invariants. Thus these developments opened a new door for the research of three-manifold invariants.

Inspired by these successes, people then tried to find other three-manifold invariants that can be expressed as weight systems whose weights depend on something else other that Lie algebra structure. This undertaking was successfully accomplished by Rozansky and Witten several years later in [10], where they formulated a certain three-dimensional supersymmetric topological sigma model with a hyperkähler target space – better known today as the Rozansky-Witten sigma model – and showed that one can, from its perturbative partition function, obtain such aforementioned three-manifold invariants whose weights depend not on Lie algebra structure but on hyperkähler geometry.

Naturally, one may further ask if there exist even more exotic three-manifold invariants that can be expressed as weight systems whose weights depend on both Lie algebra structure *and* hyperkähler geometry. Clearly, the quantum field theory relevant to this question ought to be a hybrid of the Chern-Simons theory and the Rozansky-Witten sigma model – a topological Chern-Simons sigma model if you will. Motivated by the formulation of such exotic three-manifold invariants among other things, the first example of a topological Chern-Simons sigma model – also known as the Chern-Simons-Rozansky-Witten (CSRW) sigma model – was constructed by Kapustin and Saulina in [11]. Shortly thereafter, a variety of other topological Chern-Simons sigma models was also constructed by Koh, Lee and Lee in [12], following which, the CSRW model was reconstructed via the AKSZ formalism by Källén, Qiu and Zabzine in [13], where a closely-related (albeit non-Chern-Simons) BF-Rozansky-Witten sigma model was also presented.

However, in these cited examples, the formulation and discussion of such exotic three-manifold invariants were rather abstract. To fill this gap, our main goal in this chapter is to construct an appropriate Chern-Simons sigma model

[1] that would allow us to formulate and discuss, in a concrete and down-to-earth manner accessible to most physicists, such novel and exotic three-manifold invariants, their knot generalizations, and beyond.

2.1.2 Outline

Let us now give a brief plan and summary of this chapter.

In section 2, we construct from scratch, a topological Chern-Simons sigma model on a Riemannian three-manifold M with gauge group $\mathcal{G}$ whose hyperkähler target space X is equipped with a $\mathcal{G}$-action, where $\mathcal{G}$ is a compact Lie group with Lie algebra $\mathfrak{g}$. Our model is a dynamically $\mathcal{G}$-gauged version of the Rozansky-Witten sigma model, and it is closely-related to the Chern-Simons-Rozansky-Witten sigma model of Kapustin-Saulina: the Lagrangian of the models differ only by some mass terms for certain bosonic and fermionic fields. We also present a gauge-fixed version of the action, and discuss the (in)dependence of the partition function on the various coupling constants of the theory.

In section 3, we compute perturbatively the partition function of the model. This is done by first expanding the quantum fields around points of stationary phase, and then evaluating the resulting Feynman diagram expansion of the path integral without operator insertions. Apart from obtaining *new* three-manifold invariants which define *new* weight systems whose weights are characterized by both the Lie algebra structure of $\mathfrak{g}$ *and* the hyperkähler geometry of X, we also find that (i) the one-loop contribution is a topological invariant of M that ought to be related to a hybrid of the analytic Ray-Singer torsion of the flat and trivial connection on M, respectively; (ii) an "equivariant linking number" of knots in M can be defined out of the propagators of certain fermionic fields.

In section 4, we canonically quantize the time-invariant model in a neigborhood $\Sigma \times I$ of M, where Σ is an arbitrary compact Riemann surface. We find that we effectively have a two-dimensional gauged sigma model on Σ, and that

[1] This model, just like the other CSRW-type models discussed in [11] and [12], can be constructed by topologically twisting the theories discovered by Gaiotto and Witten in [45]. The theories constructed in [45] generalize $\mathcal{N} = 4$ $d = 3$ supersymmetric gauge theories which contain a Chern-Simons gauge field interacting with $\mathcal{N} = 4$ hypermultiplets, by replacing the free hypermultiplets with a sigma model whose target space is a hyperKähler manifold.

the relevant Hilbert space of states would be given by the tensor product of the Hilbert space of Chern-Simons theory on M and the $\mathcal{G}$-equivariant cohomology of the moduli space $\mathcal{M}^{\vartheta}$ of $\mathcal{G}$-covariant maps from M to X. On three-manifolds M^U which can be obtained from M by a U-twisted surgery on $\Sigma = \mathbf{T}^2$, where U is the mapping class group of Σ, the corresponding partition function $Z_X(M^U)$ can be expressed in terms of Chern-Simons knot invariants of M and the intersection number of certain $\mathcal{G}$-equivariant cycles in $\mathcal{M}^{\vartheta}$.

In section 5, we construct supersymmetric Wilson loop operators and compute perturbatively their expectation value. In doing so, we obtain *new* knot invariants of M that also define *new* knot weight systems whose weights are characterized by both the Lie algebra structure of $\mathfrak{g}$ *and* the hyperkähler geometry of X.

2.2 A Topological Chern-Simons Sigma Model

2.2.1 The Fields and the Action

We would like to construct a topological Chern-Simons (CS) sigma model that is a dynamically $\mathcal{G}$-gauged version of the Rozansky-Witten (RW) sigma model on M with target space X, where M is a three-dimensional Riemannian manifold with local coordinates x^{μ}, $\mu = 1, 2, 3$, and X is a hyperkähler manifold of complex dimension $\dim_{\mathbb{C}} X = 2n$ which admits an action of a compact Lie group $\mathcal{G}$. Let $\{V_a\}$ where $a = 1, 2, \cdots, \dim \mathcal{G}$, be the set of Killing vector fields on X which correspond to this $\mathcal{G}$-action; they can be viewed as sections of $TX \otimes \mathfrak{g}^*$, where TX is the tangent bundle of X, while $\mathfrak{g}$ is the Lie algebra of $\mathcal{G}$. If we denote the local complex coordinates of X as $(\phi^I, \phi^{\bar{I}})$, where $I, \bar{I} = 1, \cdots, 2n$, one can also write these vector fields as

$$V_a = V_a^I \partial_I + V_a^{\bar{I}} \partial_{\bar{I}}.$$

Note that the V_a's satisfy the Lie algebra

$$[V_a, V_b] = f_{ab}^c V_c,$$

where the f^c_{ab}'s are the structure constants of $\mathfrak{g}$. Therefore, ϕ^I and $\phi^{\bar{I}}$ must transform under the $\mathcal{G}$-action as

$$\delta_\epsilon \phi^I = \epsilon^a V_a^I, \quad \delta_\epsilon \phi^{\bar{I}} = \epsilon^a V_a^{\bar{I}}.$$

In order for $\mathcal{G}$ to be a global symmetry of X, it is necessary and sufficient that (i) for all a, the V_a's are holomorphic or anti-holomorphic; (ii) the symplectic structure of X is preserved by the $\mathcal{G}$-action associated with the V_a's. If the kähler form on X is also preserved by the $\mathcal{G}$-action, locally, there would exist moment maps $\mu_+, \mu_-, \mu_3 : X \to \mathfrak{g}^*$, where

$$\mathrm{d}\mu_{+a} = -\iota_{V_a}(\Omega), \quad \mathrm{d}\mu_{-a} = -\iota_{V_a}(\bar{\Omega}), \quad \mathrm{d}\mu_{3a} = -\iota_{V_a}(J). \tag{2.1}$$

Here, $\Omega = \frac{1}{2}\Omega_{IJ}\mathrm{d}\phi^I \wedge \mathrm{d}\phi^J$ is the holomorphic symplectic form on X; $J = ig_{I\bar{K}}\mathrm{d}\phi^I \wedge \mathrm{d}\phi^{\bar{K}}$ is the kähler form on X; $g_{I\bar{K}}$ is the metric on X; and $\iota_V(\omega)$ stands for the inner product of the vector field V with the differential form ω. The moment maps μ_+, μ_-, μ_3 are assumed to exist globally (which is automatically the case if X is simply-connected), and μ_+ is holomorphic while $\mu_- = \bar{\mu}_+$ is antiholomorphic. μ_+ also satisfies

$$\{\mu_{+a}, \mu_{+b}\} = -f^c_{ab}\mu_{+c}, \tag{2.2}$$

where the curly brackets are the Poisson brackets with respect to Ω_{IJ}. Similar formulas hold for μ_- and μ_3. We further assume that X is such that

$$\mu_+ \cdot \mu_+ = \kappa^{ab}\mu_{+a}\mu_{+b} = 0, \tag{2.3}$$

because this condition is necessary for the supersymmetry transformation defined later to be nilpotent on gauge-invariant obeservables. Note that in (2.3), κ^{ab} is the inverse of the $\mathcal{G}$-invariant nondegenerate symmetric bilinear form κ_{ab} on $\mathfrak{g}$, where

$$\kappa_{ad}f^d_{bc} + \kappa_{bd}f^d_{ac} = 0. \tag{2.4}$$

Now, the fields of a $\mathcal{G}$-gauged version of the RW sigma model ought to be given by

$$\text{bosonic}: \phi^I, \phi^{\bar{I}}, A^a_\mu; \quad \text{fermionic}: \eta^{\bar{I}}, \chi^I_\mu, \tag{2.5}$$

where $I, \bar{I} = 1, \cdots, 2n$; $\mu = 1, 2, 3$; and $a = 1, \cdots \dim G$. The gauge field A is a connection one-form on a principal $\mathcal{G}$-bundle ε over M. With respect to an infinitesimal gauge transformation with parameter $\epsilon^a(x)$, it should transform as

$$\delta_\epsilon A^a = -(\mathrm{d}\epsilon^a - f^a_{bc} A^b \epsilon^c) = -D\epsilon^a. \tag{2.6}$$

Since $\mathcal{G}$ acts on X, the bosonic fields $\phi^I, \phi^{\bar{I}}$ must be sections of a fiber bundle over M associated with ε, whose typical fiber is X. Denote this bundle as X_ε. Then, the connection A also defines a nonlinear connection on X_ε where locally, it can be thought of as a one-form on M with values in the Lie algebra of vector fields on X, i.e., $A = A^a V_a$. This means that we can write the covariant differentials of ϕ^I and $\phi^{\bar{I}}$ as

$$D\phi^I = \mathrm{d}\phi^I + A^a V^I_a, \quad D\phi^{\bar{I}} = \mathrm{d}\phi^{\bar{I}} + A^a V^{\bar{I}}_a.$$

As for the fermionic fields, χ^I_μ are components of a one-form χ^I on M with values in the pullback $\phi^*(T_{X_\varepsilon})$, where T_{X_ε} is the $(1,0)$ part of the fiberwise-tangent bundle of X_ε, while $\eta^{\bar{I}}$ is a zero-form on M with values in the pullback $\phi^*(\bar{T}_{X_\varepsilon})$ of the complex-conjugate bundle $\bar{T}_{X_\varepsilon}$.

From the above expressions, it is clear that the data of the Lie group $\mathcal{G}$ and the hyperkähler geometry of X are inextricably connected. This connection will allow us to obtain *new* three-manifold invariants which depend on both $\mathcal{G}$ and X, as we will show in the next section.

The Action

With this in hand, let us now construct the action of the model. Let us assign to the fields ϕ, χ, η and A, the $U(1)$ R-charge $0, -1, 1$ and 0, respectively. Let us also define the following supersymmetry transformation of the fields under a scalar supercharge Q:

$$\begin{aligned}
&\delta_Q A_a = \chi^K \partial_K \mu_{+a},\\
&\delta_Q \phi^I = 0,\\
&\delta_Q \phi^{\bar{I}} = \eta^{\bar{I}},\\
&\delta_Q \chi^I = D\phi^I,\\
&\delta_Q \eta^{\bar{I}} = -\bar{\xi}^{\bar{I}},
\end{aligned} \tag{2.7}$$

where

$$\xi^I = V^I \cdot \mu_-, \quad \bar{\xi}^{\bar{I}} = V^{\bar{I}} \cdot \mu_+. \tag{2.8}$$

Here, the scalar supercharge Q is defined to have R-charge $+1$, while the moment maps $\mu_\pm$ are defined to have R-charge ± 2. Notice then that spin and R-charge are conserved in the above relations, as required.

From (2.8), we find that δ_Q^2 is a gauge transformation with parameter $\epsilon^a = -\kappa^{ab}\mu_{+b}$:

$$\begin{aligned}
&\delta_Q^2 A^a = \kappa^{ab}(\mathrm{d}\mu_{+b} + f^d_{cb} A^c \mu_{+d}),\\
&\delta_Q^2 \phi^I = 0, \quad \delta_Q^2 \phi^{\bar{I}} = -V^{\bar{I}} \cdot \mu_+,\\
&\delta_Q^2 \chi^I = -\chi^J \partial_J V^I \cdot \mu_+, \quad \delta_Q^2 \eta^{\bar{I}} = -\eta^{\bar{J}} \partial_{\bar{J}} V^{\bar{I}} \cdot \mu_+.
\end{aligned} \tag{2.9}$$

Note that to compute this, we have used $V^K_a \Omega_{KJ} V^J_b = f^c_{ab}\mu_{+c}$ and $V^I \cdot \mu_+ = 0$.

Thus, an example of a Q-invariant action S would be

$$\begin{aligned}
S &= \int_M (L_{cs} + L_1 + L_2),\\
L_{cs} &= \mathrm{Tr}(A \wedge dA + \frac{2}{3} A \wedge A \wedge A),\\
L_1 &= \delta_Q (g_{I\bar{K}} \chi^I \wedge \star D\phi^{\bar{K}})\\
&= g_{I\bar{K}} (D\phi^I \wedge \star D\phi^{\bar{K}} - \chi^I \wedge \star D\eta^{\bar{K}}),\\
L_2 &= \frac{1}{2}\Omega_{IJ}(\chi^I \wedge D\chi^J + \frac{1}{3} R^J_{KL\bar{M}} \chi^I \wedge \chi^K \wedge \chi^L \wedge \eta^{\bar{M}}),
\end{aligned} \tag{2.10}$$

where $\star$ denotes the Hodge star operator on differential forms on M with respect to its Riemannian metric $h_{\mu\nu}$; 'Tr' denotes a suitably-normalized invariant quadratic form on $\mathfrak{g}$; the covariant derivatives are given by

$$D\phi^I = \mathrm{d}\phi^I + A \cdot V^I, \quad D\chi^I = \nabla\chi^I + A \cdot \nabla_K V^I \chi^K, \quad D\eta^{\bar{I}} = \nabla\eta^{\bar{I}} + A \cdot \nabla_{\bar{K}} V^{\bar{I}} \eta^{\bar{K}};$$

∇ involves the Levi-Civita connection on X, where

$$\nabla\chi^I = \mathrm{d}\chi^I + \Gamma^I_{JK}\mathrm{d}\phi^J \wedge \chi^K, \quad \nabla\eta^{\bar{I}} = \mathrm{d}\eta^{\bar{I}} + \Gamma^{\bar{I}}_{\bar{J}\bar{K}}\mathrm{d}\phi^{\bar{J}} \wedge \eta^{\bar{K}},$$
$$\nabla_K V^I = \partial_K V^I + \Gamma^I_{KJ} V^J, \quad \nabla_{\bar{K}} V^{\bar{I}} = \partial_{\bar{K}} V^{\bar{I}} + \Gamma^{\bar{I}}_{\bar{K}\bar{J}} V^{\bar{J}};$$

and $R^J_{KL\bar{M}}$ denotes the curvature tensor of the Levi-Civita connection on X, where

$$R^J_{KL\bar{M}} = \frac{\partial \Gamma^J_{KL}}{\partial \phi^{\bar{M}}}, \quad \Gamma^I_{JK} = (\partial_J g_{K\bar{M}}) g^{I\bar{M}}.$$

Gauge-Fixing

One of our main objectives in this chapter is to compute the partition function of the model. To do so, we need to gauge-fix the model. This can be done as follows.

Define the total BRST transformation

$$\delta_{\hat{Q}} = \delta_Q + \delta_{FP},$$

where δ_{FP} is the usual Faddeev-Popov BRST operator with R-charge $+1$. The total BRST transformation $\delta_{\hat{Q}}$ must be nilpotent, while δ_Q is nilpotent only up to a gauge transformation.

We then extend the theory by introducing fermionic Faddev-Popov ghost and anti-ghost fields c^a, $\bar{c}_a$, as well as bosonic Lagrangian multiplier fields B_a. c, $\bar{c}$, B are defined to have R-charge 1, -1 and 0, respectively. c takes values in $\mathfrak{g}$, while $\bar{c}$ and B take values in the dual Lie algebra $\mathfrak{g}^*$. By conservation of spin and R-charge, the total BRST operator $\hat{Q}$ should act on the fields as

$$\begin{aligned}
\delta_{\hat{Q}} A_a &= \mathrm{d}c_a - f_{abd}A^b c^d + \chi^K \partial_K \mu_{+a}, \\
\delta_{\hat{Q}} \phi^I &= -V^I \cdot c, \\
\delta_{\hat{Q}} \phi^{\bar{I}} &= \eta^{\bar{I}} - V^{\bar{I}} \cdot c, \\
\delta_{\hat{Q}} \chi^I &= D\phi^I + (\partial_J V^{Ia})\chi^J c_a, \\
\delta_{\hat{Q}} \eta^{\bar{I}} &= \bar{\xi}^{\bar{I}} + (\partial_{\bar{J}} V^{\bar{I}a})\eta^{\bar{J}} c_a, \\
\delta_{\hat{Q}} c^a &= -\kappa^{ab}\mu_{+b} + \frac{1}{2} f^a_{bc} c^b c^c, \\
\delta_{\hat{Q}} \bar{c} \ &= B, \\
\delta_{\hat{Q}} B &= 0.
\end{aligned} \tag{2.11}$$

It's easy to show that $\delta_{\hat{Q}}^2 = 0$ on the fields. The $\hat{Q}$-invariant gauge-fixed action S would then be

$$\begin{aligned}
S &= \int_M (L_{cs} + L_1 + L_2), \\
L_{cs} &= \mathrm{Tr}(A \wedge dA + \frac{2}{3} A \wedge A \wedge A), \\
L_1 &= \delta_{\hat{Q}}(g_{I\bar{K}} \chi^I \wedge \star D\phi^{\bar{K}} + \bar{c}_a f^a) \\
&= g_{I\bar{K}}(D\phi^I \wedge \star D\phi^{\bar{K}} - \chi^I \wedge \star D\eta^{\bar{K}}) + B_a f^a - \bar{c}_a \delta_{\hat{Q}} f^a, \\
L_2 &= \frac{1}{2}\Omega_{IJ}(\chi^I \wedge D\chi^J + \frac{1}{3} R^J_{KL\bar{M}} \chi^I \wedge \chi^K \wedge \chi^L \wedge \eta^{\bar{M}}),
\end{aligned} \tag{2.12}$$

where f^a is some $\mathfrak{g}$-valued function; $\int_M L_{cs}$ and $\int_M L_2$ are manifestly independent of the metric of M; while $L_1 = \{\hat{Q}, \dots\}$ is an exact form of the total BRST operator $\hat{Q}$. Since the metric dependence of the action is of the form $\{\hat{Q}, \dots\}$, the partition function, and also the correlation functions of $\hat{Q}$-closed operators, are metric independent. In this sense, the theory is topologically invariant.

Notice that the transformation on the ghost field c is not standard. The standard ghost field transformation just involves the usual δ_{FP} variation, while c also gets transformed by δ_Q:

$$\delta_Q c^a = \kappa^{ab}\mu_{+b}. \tag{2.13}$$

This fact makes the part of the action involving ghost and anti-ghost fields non-standard. For example, if we choose the Lorentz gauge $f^a = \partial^\mu A^a_\mu$, the action contains the term $\bar{c}^a \partial^\mu(\chi^K \mu_{+a})$ where the anti-ghost field $\bar{c}^a$ is coupled to the 'matter' fermion χ^K.

2.2.2 About the Coupling Constants

Before we end this section, let us discuss the coupling constants of the theory as it would prove useful to do so when we carry out our computation of the partition function and beyond in the rest of the chapter.

To this end, note that the partition function can be written as

$$Z = \int D\phi D\eta D\chi DADcD\bar{c}DB\ \exp\left(-\int_M (k_{cs}L_{cs} + k_1 L_1 + k_2 L_2)\sqrt{h}\mathrm{d}^3x\right), \tag{2.14}$$

where k_1, k_2 and k_{cs} are the possible coupling constants of the theory. As

$$\frac{\delta Z}{\delta k_1} = \langle \delta_{\hat{Q}} \mathcal{O} \rangle = 0, \tag{2.15}$$

the partition function should not depend on k_1.

Let us now rescale the fields as follows:

$$\eta \to \lambda\eta, \quad \chi \to \lambda^{-1}\chi, \quad \bar{c} \to \lambda\bar{c}, \quad c \to \lambda^{-1}c, \tag{2.16}$$

whence

$$k_1 L_1 \to k_1 L_1, \quad k_2 L_2 \to \lambda^{-2} k_2 L_2. \tag{2.17}$$

As the field rescaling should not change the theory, the partition function should not depend on k_2 either. Thus, let us just write

$$k_1 = k_2 = k. \tag{2.18}$$

That being said, our partition function *does* depend on the coupling constant k_{cs}. Moreover, because of the requirement of gauge invariance [3], k_{cs} ought to be quantized as

$$k_{cs} = \frac{m}{2\pi}; \quad m = 1, 2, 3 \ldots \tag{2.19}$$

Hence, we have *two* physically distinct coupling constants in our theory. This should come as no surprise since our theory is actually a combination of a Schwarz- and Witten-type topological field theory.

2.3 The Perturbative Partition Function and New Three-Manifold Invariants

2.3.1 The Perturbative Partition Function

Let us now proceed to discuss the partition function of the gauged sigma model in the perturbative limit. To this end, recall from the last section that the partition function depends on the coupling k_{cs}. Hence, the perturbative limit of the (CS part of the) model is the same as its large k_{cs} limit. Moreover, because the partition function is independent of k, we can choose $k_1 = k_2 = k$ as large as we want. Altogether, this means that the perturbative partition function would be given by a sum of contributions centered around the points of stationary phase characterized by

$$\frac{\delta L_{cs}}{\delta A} = \mathrm{d}A + [A, A] = 0, \tag{2.20}$$

which are the *flat connections*, and

$$\frac{\delta S}{\delta \phi} = 0 \to D^{\mu}\phi = 0, \tag{2.21}$$

which are the *covariantly constant* maps from M to X.

Thus, where the perturbative partition function is concerned, we can expand the gauge field A around the flat connection A_0^{ϑ} as

$$A_{\mu}^{a}(x) = A_{0\mu}^{\vartheta a}(x) + \tilde{A}_{\mu}^{a}(x), \tag{2.22}$$

and the bosonic scalar fields ϕ around the covariantly constant map ϕ_0 as

$$\phi^{I}(x) = \phi_0^{I}(x) + \varphi^{I}(x), \quad \phi^{\bar{I}}(x) = \phi_0^{\bar{I}}(x) + \varphi^{\bar{I}}(x), \tag{2.23}$$

where

$$D^\mu \phi_0^I = \partial^\mu \phi_0^I + A_0^{\vartheta a\mu} V_a^I(\phi_0) = 0, \qquad D^\mu \phi_0^{\bar{I}} = \partial^\mu \phi_0^{\bar{I}} + A_0^{\vartheta a\mu} V_a^{\bar{I}}(\phi_0) = 0. \quad (2.24)$$

Note that (2.22) means that we can write

$$L_{cs} = L_{cs}(A_0^\vartheta) + \tilde{A} \wedge \mathrm{d}\tilde{A} + \tilde{A} \wedge [A_0^\vartheta, \tilde{A}] + \frac{2}{3}\tilde{A} \wedge \tilde{A} \wedge \tilde{A}. \qquad (2.25)$$

Let $\mathcal{M}^\vartheta$ be the space of physically distinct ϕ_0's which satisfy (2.24) for some flat connection A_0^ϑ. Assuming that the flat connection A_0^ϑ is isolated,[2] we can then write our perturbative partition function as

$$Z = k^{2n} \sum_{A_0^\vartheta} \left(e^{-\int_M L_{cs}(A_0^\vartheta)} \int_{\mathcal{M}^\vartheta} \prod_{I=1}^{2n} \mathrm{d}\phi_0^I \prod_{\bar{I}=1}^{2n} \mathrm{d}\phi_0^{\bar{I}} \int D\varphi D\chi D\eta D\tilde{A} Dc D\bar{c} DB \, e^{-S_{A_0^\vartheta,\phi_0}} \right). \qquad (2.26)$$

Here, k^{2n} is the normalization factor carried by the $2n$ bosonic zero modes ϕ_0, and $\int_M L_{cs}(A_0^\vartheta) + S_{A_0^\vartheta,\phi_0}$ is the total action expanded around A_0^ϑ and ϕ_0.

In the total action expanded around the flat gauge field A_0^ϑ and the covariantly constant bosonic scalar fields $\phi_0^{I,\bar{I}}$, we have

$$\begin{aligned} D_\mu \phi^I &= \partial_\mu \phi_0^I + \partial_\mu \varphi^I + (A_{0\mu}^{\vartheta a} + \tilde{A}_\mu^a)\{V_a^I(\phi_0) + \varphi^J \partial_J V_a^I(\phi_0) + \frac{\varphi^J\varphi^K}{2}\partial_J\partial_K V_a^I(\phi_0) + \cdots\} \\ &= \partial_\mu \varphi^I + A_{0\mu}^{\vartheta a}\varphi^J \partial_J V_a^I + \tilde{A}_\mu^a V_a^I + \tilde{A}_\mu^a \varphi^J \partial_J V_a^I + (A_{0\mu}^{\vartheta a} + \tilde{A}_\mu^a)(\frac{\varphi^J\varphi^K}{2}\partial_J\partial_K V_a^I) + \cdots, \end{aligned} \qquad (2.27)$$

since $D_\mu \phi_0^I = \partial_\mu \phi_0^I + A_{0\mu}^{\vartheta a} V_a^I(\phi_0) = 0$. We also have

$$\begin{aligned} D_\mu \chi_\nu^I &= \partial_\mu \chi_\nu^I + \partial_\mu \phi^J \Gamma_{JK}^I \chi_\nu^K + A_\mu^a \nabla_J V_a^I \chi_\nu^J \\ &= \partial_\mu \chi_\nu^I + \Gamma_{JK}^I D_\mu \phi^J \chi_\nu^K + A_\mu^a \partial_J V_a^I \chi_\nu^J \\ &= \partial_\mu \chi_\nu^I + \Gamma_{JK}^I D_\mu \phi^J \chi_\nu^K + (A_{0\mu}^{\vartheta a} + \tilde{A}_\mu^a)(\partial_J V_a^I + \partial_K \partial_J V_a^I \varphi^K + \cdots)\chi_\nu^J, \end{aligned} \qquad (2.28)$$

where $D_\mu \phi^J$ is as given in (2.27). Similarly, one can compute the expansion of $D_\mu \eta^{\bar{I}}$.

[2]This would indeed be the case if $H^1(M, E) = 0$, where E is a flat bundle determined by A_0.

Because of (2.27) and (2.28), we can rewrite our Lagrangian as

$$L = \text{quadratic part} + \text{vertices}, \tag{2.29}$$

where

$$\text{quadratic part} = k \begin{pmatrix} \varphi^I & \varphi^{\bar{I}} & \tilde{A}^a_\mu \end{pmatrix} \cdot L_{\text{boson}} \cdot \begin{pmatrix} \varphi^J \\ \varphi^{\bar{J}} \\ \tilde{A}^b_\rho \end{pmatrix} + k(\; \eta^{\bar{I}} \quad \chi^I_\mu \quad \bar{c}^a \quad c^a \;) \cdot L_{\text{fermion}} \cdot \begin{pmatrix} \eta^{\bar{J}} \\ \chi^J_\rho \\ \bar{c}^b \\ c^b \end{pmatrix}, \tag{2.30}$$

$$\begin{aligned} \text{vertices} =& \frac{k_{cs}}{3} \epsilon^{\mu\nu\rho} f_{abc} \tilde{A}^a_\mu \tilde{A}^b_\nu \tilde{A}^c_\rho \\ & - \frac{k}{2} g_{I\bar{K}} \left(\Gamma^{\bar{K}}_{\bar{M}\bar{N}} \partial_\mu \varphi^{\bar{M}} \chi^{I\mu} \eta^{\bar{N}} + \nabla_{\bar{P}} V^{\bar{K}a} \eta^{\bar{P}} \chi^I_\mu \tilde{A}^\mu_a \right) \\ & + \frac{k}{2\sqrt{h}} \Omega_{IJ} \epsilon^{\mu\nu\rho} \left(\Gamma^J_{MN} \partial_\nu \varphi^M \chi^I_\mu \chi^N_\rho + \nabla_P V^{Ja} \chi^I_\mu \chi^P_\rho \tilde{A}_{\nu a} \right) \\ & + \frac{k}{6\sqrt{h}} \Omega_{IJ} \epsilon^{\mu\nu\rho} R^J_{KL\bar{M}} \chi^I_\mu \chi^K_\nu \chi^L_\rho \eta^{\bar{M}} + f_{abd} \left(\bar{c}^a \partial^\mu \tilde{A}^b_\mu c^d + \bar{c}^a \tilde{A}^b_\mu \partial^\mu c^d \right) \\ & - \frac{k}{2} \partial_K \Gamma_{I\bar{M}\bar{N}} \partial_\mu \varphi^{\bar{M}} \varphi^K \chi^{I,\mu} \eta^{\bar{N}} + \cdots . \end{aligned}$$

Here,

$$L_{\text{boson}} = \begin{pmatrix} 0 & L^{\varphi\varphi}_{I\bar{J}} & -\frac{1}{2} V_{Ib} \partial^\rho + A^{\vartheta\rho}_{0b} V_{Ka} \partial_I V^{Kb} \\ L^{\varphi\varphi}_{\bar{I}J} & 0 & -\frac{1}{2} V_{\bar{I}b} \partial^\rho + A^{\vartheta\rho}_{0b} V_{\bar{K}a} \partial_{\bar{I}} V^{\bar{K}b} \\ \frac{1}{2} V_{Ja} \partial^\mu + A^{\vartheta\mu}_{0b} V_{Ka} \partial_J V^{Kb} & \frac{1}{2} V_{\bar{J}a} \partial^\mu + A^{\vartheta\mu}_{0b} V_{\bar{K}a} \partial_{\bar{J}} V^{\bar{K}b} & V_{Ka} V^K_b \delta^{\mu\rho} + \frac{k_{cs}}{k} \epsilon^{\mu\nu\rho} (\kappa_{ab} \partial_\nu + \frac{1}{3} f_{adb} A^{\vartheta d}_{0\nu}) \end{pmatrix}, \tag{2.31}$$

where

$$\begin{aligned} L^{\varphi\varphi}_{I\bar{J}} &:= \frac{1}{2} (g_{I\bar{J}} \partial^2 + g_{K\bar{L}} A^{\vartheta a}_{0\mu} A^{\vartheta\mu b}_0 \partial_I V^K_a \partial_{\bar{J}} V^{\bar{L}}_b) + g_{P\bar{J}} \partial_I V^P_a A^{\vartheta a}_{0\mu} \partial^\mu, \\ L^{\varphi\varphi}_{\bar{I}J} &:= \frac{1}{2} (g_{\bar{I}J} \partial^2 + g_{K\bar{L}} A^{\vartheta a}_{0\mu} A^{\vartheta\mu b}_0 \partial_J V^K_a \partial_{\bar{I}} V^{\bar{L}}_b) + g_{J\bar{P}} \partial_{\bar{I}} V^{\bar{P}}_a A^{\vartheta a}_{0\mu} \partial^\mu; \end{aligned} \tag{2.32}$$

$$
L_{\text{fermion}} = \begin{pmatrix} 0 & -\frac{1}{2}(g_{\bar{I}J}\partial^{\rho} + g_{J\bar{K}}\partial_{\bar{I}}V_a^{\bar{K}}A_0^{\vartheta a\rho}) & 0 & 0 \\ \frac{1}{2}(g_{I\bar{J}}\partial^{\mu} + g_{I\bar{K}}\partial_{\bar{J}}V^{\bar{K}a}A_{0a}^{\vartheta\mu}) & \frac{\Omega_{IJ}}{2\sqrt{h}}\epsilon^{\mu\nu\rho}\partial_{\nu} + \frac{\Omega_{IK}}{2\sqrt{h}}\epsilon^{\mu\nu\rho}\partial_J V_a^K A_{0\nu}^{\vartheta a} & -\frac{1}{2}\partial_I\mu_{+b}\partial^{\mu} & 0 \\ 0 & \frac{1}{2}\partial_J\mu_{+a}\partial^{\rho} & 0 & -\frac{1}{2}\delta_b^a\partial^2 \\ 0 & 0 & \frac{1}{2}\delta_b^a\partial^2 & 0 \end{pmatrix}; \tag{2.33}
$$

the terms g_{IK} and $\Gamma^{\bar{K}}_{\bar{M}\bar{N}}$ are evaluated at ϕ_0; and $\cdots$ are other terms expanded around ϕ_0. Also notice that we ignore the multiplier field B in L_{boson}, for it would just be integrated out to give the gauge-fixing condition $f^a = 0$, where we have chosen the gauge

$$
f^a = \partial^{\mu}A_{\mu}^a = 0. \tag{2.34}
$$

We can further separate the integration over the fermion zero modes η_0 and χ_0 in the path-integral and write[3]

$$
Z = k^{2n}\sum_{A_0^{\vartheta}}\left(e^{-\int_M L_{cs}(A_0^{\vartheta})}\int_{\mathcal{M}^{\vartheta}}\prod_{I=1}^{2n}d\phi_0^I\prod_{\bar{I}=1}^{2n}d\phi_0^{\bar{I}}\prod_{\bar{I}_i=1}^{2nb_0'}\prod_{I_j=1}^{2nb_1'}\int d\eta_0^{\bar{I}_i}\int d\chi_{0\mu}^{I_j}\int D\varphi D\tilde{\chi}D\tilde{\eta}D\tilde{A}DcD\bar{c}\,e^{-S_{A_0^{\vartheta},\phi_0}}\right), \tag{2.35}
$$

where b_0' and b_1' denote the number of fermionic zero modes $\eta_0^{\bar{I}}$ and χ_0^I, respectively; $\tilde{\eta}$ and $\tilde{\chi}$ are the corresponding *nonzero* modes; and k^{2n} is the normalization factor carried by the bosonic zero modes. One should note that the fermionic zero modes $\eta_0^{\bar{I}}$ and χ_0^I are no longer harmonic forms on M like in RW theory; this is because in our case, the kinetic operator of the fermionic fields L_{fermion} in (2.33) is no longer the Laplacian operator but a covariant version thereof. In the limit $A \to 0$, b_0', b_1' become the respective Betti numbers of M, while (2.35) becomes the partition function of the RW theory.

2.3.2 One-Loop Contribution

As usual, the one-loop contribution to the perturbative partition function is given by

$$
Z_0 = \int D\varphi D\tilde{\chi}D\tilde{\eta}DcD\bar{c}\; e^{-S_0}, \tag{2.36}
$$

[3]Here, in addition to footnote 2, we assume that $H^0(M,E) = 0$ so that there are no zero modes for the ghost fields c, $\bar{c}$ and B.

where S_0 is quadratic in the fluctuating bosonic fields $\{\tilde{A}_a^\mu, \varphi^i(x)\}$ and the fermionic *nonzero* modes $\{\tilde{\eta}^I, \tilde{\chi}_\mu^I\}$:

$$S_0 = \int_M k(\ \varphi^I \quad \varphi^{\bar{I}} \quad \tilde{A}_\mu^a\ ) \cdot L_{\mathrm{boson}} \cdot \begin{pmatrix} \varphi^J \\ \varphi^{\bar{J}} \\ \tilde{A}_\rho^b \end{pmatrix} + k(\ \tilde{\eta}^{\bar{I}} \quad \tilde{\chi}_\mu^I \quad \bar{c}^a \quad c^a\ ) \cdot L_{\mathrm{fermion}} \cdot \begin{pmatrix} \tilde{\eta}^{\bar{J}} \\ \tilde{\chi}_\rho^J \\ \bar{c}^b \\ c^b \end{pmatrix}. \tag{2.37}$$

Here, the tensors $g_{I\bar{J}}, \Omega_{IJ}$ and Γ^I_{JK} which appear in L_{boson} and L_{fermion} are evaluated at some ϕ_0 in $\mathcal{M}^\vartheta$.

To compute (2.36), we first diagonalize L_{boson} and L_{fermion}:

$$\begin{aligned} P_B^T \cdot L_{\mathrm{boson}} \cdot P_B &= L'_{\mathrm{boson}}, \\ P_F^T \cdot L_{\mathrm{fermion}} \cdot P_F &= L'_{\mathrm{fermion}}, \end{aligned} \tag{2.38}$$

where L'_{boson} and L'_{fermion} are diagonal matrices, and P_B and P_F are orthonormal matrices ($P^T = P^{-1}$) constructed from the eigenvectors of L_{boson} and L_{fermion}. Because $PP^T = 1$, we can rewrite S_0 as

$$S'_0 = \int_M k(\ \varphi'^I \quad \varphi'^{\bar{I}} \quad \tilde{A}'^a_\mu\ ) \cdot L'_{\mathrm{boson}} \cdot \begin{pmatrix} \varphi'^J \\ \varphi'^{\bar{J}} \\ \tilde{A}'^b_\rho \end{pmatrix} + k(\ \tilde{\eta}'^{\bar{I}} \quad \tilde{\chi}'^I_\mu \quad \bar{c}'^a \quad c'^a\ ) \cdot L'_{\mathrm{fermion}} \cdot \begin{pmatrix} \tilde{\eta}'^{\bar{J}} \\ \tilde{\chi}'^J_\rho \\ \bar{c}'^b \\ c'^b \end{pmatrix}, \tag{2.39}$$

where

$$\begin{pmatrix} \varphi'^J \\ \varphi'^{\bar{J}} \\ \tilde{A}'^b_\rho \end{pmatrix} := P_B^T \begin{pmatrix} \varphi^J \\ \varphi^{\bar{J}} \\ \tilde{A}^b_\rho \end{pmatrix}, \tag{2.40}$$

and

$$\begin{pmatrix} \tilde{\eta}'^{\bar{J}} \\ \tilde{\chi}'^J_\rho \\ \bar{c}'^b \\ c'^b \end{pmatrix} := P_F^T \begin{pmatrix} \tilde{\eta}^{\bar{J}} \\ \tilde{\chi}^J_\rho \\ \bar{c}^b \\ c^b \end{pmatrix}. \tag{2.41}$$

Moreover, the Jocabian determinants

$$\det(P) = \det(P^T) = 1 \tag{2.42}$$

for both the bosonic and fermionic fields. Therefore, the measure of the path integral is such that

$$D\varphi D\tilde{\chi} D\tilde{\eta} Dc D\bar{c} = D\varphi' D\tilde{\chi}' D\tilde{\eta}' Dc' D\bar{c}'. \tag{2.43}$$

In all, this means that the one-loop partition function can be rewritten as

$$Z_0 = \int D\varphi' D\tilde{\chi}' D\tilde{\eta}' Dc' D\bar{c}' e^{S_0'}. \tag{2.44}$$

Now because L'_{boson} and L'_{fermion} are diagonal matrices, the path integral becomes a Gaussian integral which can be directly computed as

$$Z_0 = \left(\frac{\det'' L'_{\text{fermion}}}{\det'' L'_{\text{boson}}} \right)^{\frac{1}{2}}, \tag{2.45}$$

where by using (2.38) and $\det(P^T P) = 1$, we finally get

$$Z_0 = \left(\frac{\det'' L_{\text{fermion}}}{\det'' L_{\text{boson}}} \right)^{\frac{1}{2}}. \tag{2.46}$$

Here, the superscript $''$ indicates that only nonzero modes are considered, and L_{fermion} and L_{boson} are explicitly given by (2.33) and (2.31), respectively.

As discussed in [7], the (magnitude of the) one-loop contribution to the perturbative partition function of CS theory on M corresponds to the analytic Ray-Singer torsion of the flat connection on M, while the (magnitude of the) one-loop contribution to the perturbative partition function of RW theory on M corresponds to the analytic Ray-Singer torsion of the trivial connection on M. Since our theory is a combination of both these theories, (the magnitude of) Z_0 ought to be related to a hybrid of these aforementioned topological invariants of M.

2.3.3 The Vacuum Expectation Value of Fermionic Zero Modes

Notice that we may call the zero modes $\chi_{0\mu}^I$ and $\eta_0^{\bar{I}}$ of the covariant Laplacian operator L_{fermion}, covariant harmonic one- and zero-forms on M with values in the tangent and complex-conjugate tangent fibres $V_{\phi_0(x)}$ and $\bar{V}_{\phi_0(x)}$ over $\mathcal{M}^\vartheta$ evaluated at the covariantly constant map $\phi_0(x)$. Because

$$\begin{aligned}
\#(\text{zero modes of } \eta^{\bar{I}}) &= 2n \times b_0^{'}, \\
\#(\text{zero modes of } \chi_\mu^I) &= 2n \times b_1^{'},
\end{aligned} \tag{2.47}$$

only a product of $2nb_0^{'}$ fields η_0^I with $2nb_1^{'}$ fields $\chi_{0\mu}^I$ has a nonzero vacuum expectation value.

Notice also that the self-products of $\eta_0^{\bar{I}}$ and $\chi_{0\mu}^I$ are elements of the space

$$\begin{aligned}
H_\eta &= \wedge^{\max}(\Omega^{'0}(M) \otimes \bar{V}_{\phi_0(x)}), \\
H_\chi &= \wedge^{\max}(\Omega^{'1}(M) \otimes V_{\phi_0(x)}),
\end{aligned} \tag{2.48}$$

where $\Omega^{'i}(M)$ is the space of covariant harmonic i-forms on M. There is a lattice inside $\Omega^{'1}(M)$ which is formed by covariant harmonic one-forms with integer-valued integrals over dual one-cycles in M; let $\omega_\mu^{(\alpha)}$, where $1 \leq \alpha \leq b_1^{'}$, be a basis of this lattice. Then, a natural measure for the fermion zero modes can be defined by normalizing the fermionic vacuum expectation values as

$$\langle \eta_0^{\bar{I}_1}(x_1) \cdots \eta_0^{\bar{I}_{2nb_0^{'}}}(x_{2nb_0^{'}}) \rangle = k^{-nb_0^{'}} \epsilon^{\bar{I}_1 \cdots \bar{I}_{2nb_0^{'}}} := \frac{k^{-nb_0^{'}}}{2n} \sum_{s \in S_{2nb_0^{'}}} (-1)^{|s|} \epsilon^{\bar{I}_{s(1)} \bar{I}_{s(2)}} \cdots \epsilon^{\bar{I}_{s(2nb_0^{'}-1)} \bar{I}_{s(2nb_0^{'})}}, \tag{2.49}$$

and

$$\begin{aligned}
\langle \chi_{0\mu_1}^{I_1}(x_1) \cdots \chi_{0\mu_{2nb_1^{'}}}^{I_{2nb_1^{'}}}(x_{2nb_1^{'}}) \rangle &= \frac{k^{-nb_1^{'}}}{((2n)!)^{b_1^{'}}} \sum_{s \in S_{2nb_1^{'}}} (-1)^{|s|} \times \\
&\prod_{\alpha=0}^{b_1^{'}-1} \left(\epsilon^{I_{s(2\alpha n+1)} \cdots I_{s(2\alpha n+2n)}} \omega_{\mu_{s(2\alpha n+1)}}^{(\alpha)}(x_{s(2\alpha n+1)}) \cdots \omega_{\mu_{s(2\alpha n+2n)}}^{(\alpha)}(x_{s(2\alpha n+2n)}) \right),
\end{aligned} \tag{2.50}$$

say in the Feynman diagrams associated with the computation of the perturbative partition function, where S_m is the symmetric group of m elements, and $|s|$ is the parity of a permutation s.

Analogous to RW theory, a choice of an overall sign in (2.49) and (2.50) for the fermionic expectation values, is equivalent to a choice of orientations on the spaces

$$\Omega'^0(M) \otimes \bar{V}_{\phi_0(x)}, \quad \Omega'^1(M) \otimes V_{\phi_0(x)}. \tag{2.51}$$

As a result, the whole partition function Z is an invariant of M up to a choice of orientation on the spaces (2.51), as the sign of Z depends on this choice.

Note that the orientations of the spaces $\bar{V}_{\phi_0(x)}$ and $V_{\phi_0(x)}$ are determined by the nth power of the two-forms $\epsilon_{\bar{I}\bar{J}}$ and ϵ_{IJ} on $\mathcal{M}^\vartheta$, respectively. On the other hand, since $\bar{V}_{\phi_0(x)}$ and $V_{\phi_0(x)}$ are both even-dimensional, the orientation on the spaces (2.51) does not depend on the choice of orientation on the spaces $\Omega'^0(M)$ and $\Omega'^1(M)$, and this is why the sign of the expectation value (2.50) does not depend on the choice of covariant harmonic one-forms $\omega_\mu^{(\alpha)}$. Therefore, the choice of orientation of the spaces (2.51) and consequently, the choice of the sign in (2.49) and (2.50), can always be reduced to a canonical orientation.

In discussing this orientation dependency, we have followed the analysis in [10]. This is because in the spaces (2.51), $\Omega'^0(M)$, $\Omega'^1(M)$ and $\mathcal{M}^\vartheta$ (the base space for the fibres $\bar{V}_{\phi_0(x)}$ and $V_{\phi_0(x)}$), are just covariant versions of the harmonic forms and space of constant bosonic maps considered in RW theory, whence the analysis would be the same.

2.3.4 Feynman Diagrams

Let us now analyze the Feynman diagrams associated with the computation of the perturbative partition function. Note that all diagrams which contribute to the partition function should have (i) the right number of fermionic zero modes in the corresponding vertices to absorb those that appear in the path integral measure; (ii) a k^{-2n} factor for canceling the normalization factor k^{2n} that accompanies the partition function in (2.35), because the partition function should be independent of the coupling constant k.

In RW theory [10], only a finite number of diagrams contribute to the partition function after (i) and (ii) are satisfied. In our case however, because we have, in our action, a Chern-Simons part with coupling constant $k_{cs} \neq k$, there would be an infinite number of diagrams contributing to our partition function. Fortunately though, the analysis is still tractable whence we would be able to derive some very insightful and concrete formulas in the end, as we shall see.

Canceling the Normalization Factor of k^{2n}

At any rate, before we proceed to say more about the Feynman diagrams, let us discuss how one can cancel the aforementioned normalization factor of k^{2n}. To this end, first note that in the CS part of the action, the gauge field has quadratic term

$$k_{cs} A D^0 A = k_{cs} \epsilon^{\mu\nu\rho} A^a_\mu (\kappa_{ab} \partial_\rho + \frac{1}{3} f_{adb} A^{\vartheta d}_{0\rho}) A^b_\nu. \tag{2.52}$$

Therefore, the propagator of the gauge field is *a priori*

$$\triangle^{A_\mu A_\nu} \sim \frac{1}{k_{cs}}. \tag{2.53}$$

However, upon expanding the Lagrangian around A_0 and ϕ_0, the gauge field will acquire a mass term

$$k V_{Ka} V^K_b A^{\nu a} A^b_\nu. \tag{2.54}$$

As such, the propagator would become

$$\triangle^{A_\mu A_\nu} = (k_{cs} D^0 + k V_K \cdot V^K)^{-1}. \tag{2.55}$$

That being said, because the partition function does not depend on k, we can choose

$$\frac{k_{cs}}{k} \gg 1. \tag{2.56}$$

In turn, this means from (2.55) that

$$\triangle^{A_\mu A_\nu} \sim \frac{1}{k_{cs}}. \tag{2.57}$$

Hence, in what follows, we will note that $\triangle^{A_\mu A_\nu} \sim k_{cs}^{-1}$, while the other propagators are $\sim k^{-1}$.

Now, let us consider a diagram with V vertices, emanating L legs. Assume that this diagram contains V_{cs} vertices $\frac{k_{cs}}{3} A \wedge A \wedge A$ which therefore contribute a factor of $k_{cs}^{V_{cs}}$; all the other $V - V_{cs}$ vertices therefore contribute a factor of $k^{V-V_{cs}}$. Let L_{cs} be the total number of legs which are joined together by the propagator $\triangle^{A_\mu A_\nu}$, where $\mu \neq \nu$; they contribute a factor of $k_{cs}^{-\frac{L_{cs}}{2}}$. As the other propagators carry a factor of k^{-1}, while each fermionic zero mode carries a normalization factor of $k^{-\frac{1}{2}}$, the remaining $L - L_{cs}$ legs contribute a factor of $k^{-\frac{L-L_{cs}}{2}}$. Thus, this diagram contains a factor of

$$k^{-\left(\frac{L-L_{cs}}{2} - (V - V_{cs})\right)}, \tag{2.58}$$

but because the partition function is independent of k, it must be that

$$\frac{L - L_{cs}}{2} - (V - V_{cs}) = 2n. \tag{2.59}$$

In other words, our diagrams must obey (2.59) so that the normalization factor of k^{2n} can be cancelled out.

Notice that in the case where $A \to 0$ whence $L_{cs} = V_{cs} = 0$ and our model reduces to the RW model, (2.59) would coincide with [10, eqn. (3.25)], as expected.

The Structure of the Feynman Diagrams

Note that although the computation of the partition function involves summing an infinite number of Feynman diagrams because there is no constraint on k_{cs}, one can actually classify the vertices they involve into three types.

(1) The pure gauge field vertex coming from the CS interaction

$$k_{cs} f_{abc} \epsilon^{\mu\nu\rho} A^a_\mu A^b_\nu A^c_\rho. \tag{2.60}$$

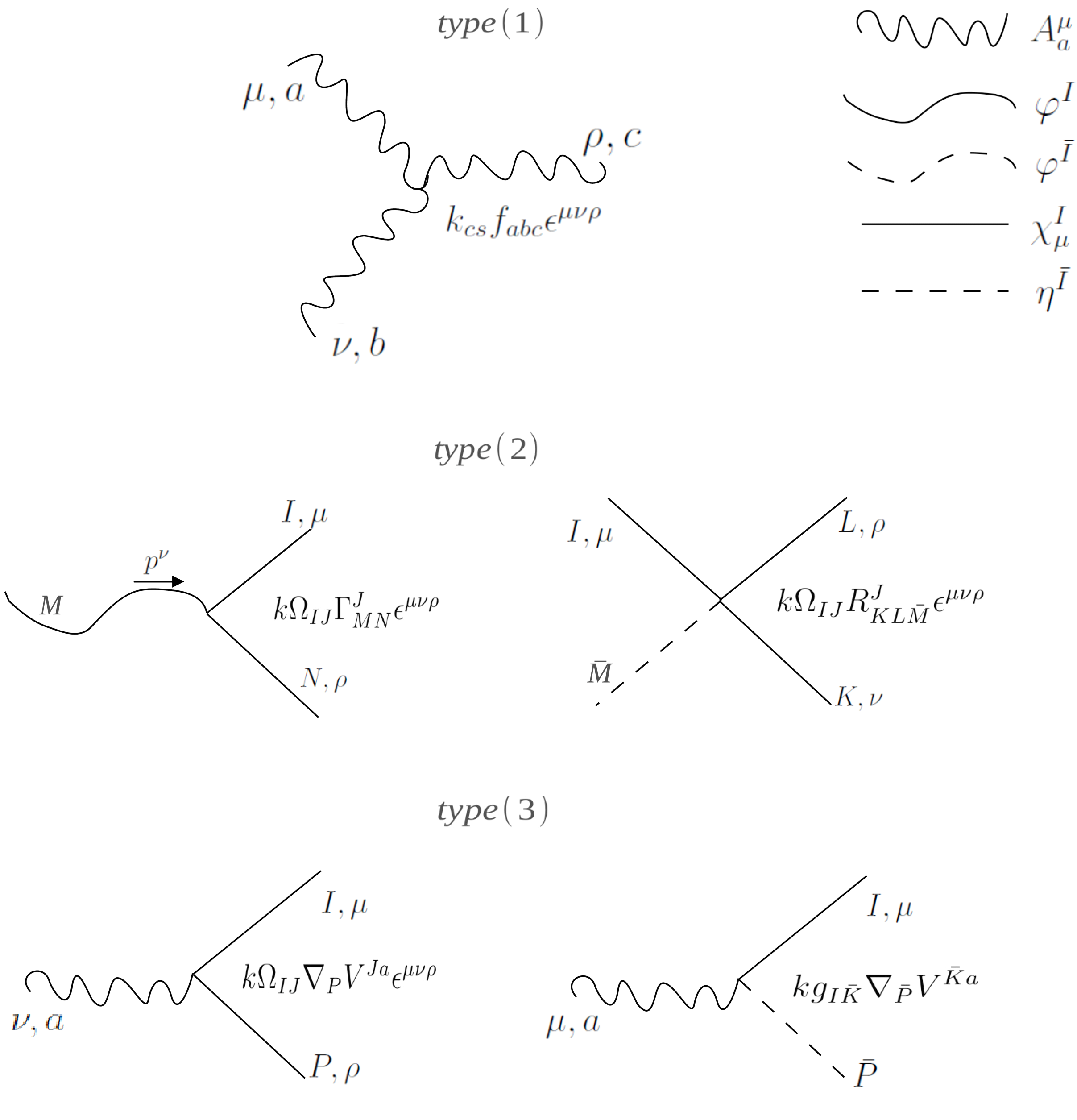

FIGURE 2.1: The Three Types of Vertices

(2) The vertices free of gauge fields, such as

$$k\Omega_{IJ}\epsilon^{\mu\nu\rho}\Gamma^J_{MN}\partial_\nu\varphi^M\chi^I_\mu\chi^N_\rho \qquad \text{and} \qquad k\Omega_{IJ}\epsilon^{\mu\nu\rho}R^J_{KL\bar{M}}\chi^I_\mu\chi^K_\nu\chi^L_\rho\eta^{\bar{M}}. \tag{2.61}$$

(3) The vertices that mix matter fields[4] with gauge fields, such as

$$k\Omega_{IJ}\epsilon^{\mu\nu\rho}\nabla_P V^{Ja}\chi^I_\mu\chi^P_\rho A_{\nu a} \qquad \text{and} \qquad kg_{I\bar{K}}\nabla_{\bar{P}}V^{\bar{K}a}\eta^{\bar{P}}\chi^I_\mu A^\mu_a. \tag{2.62}$$

These three types of vertices are illustrated in Fig. 1.

2.3.5 The Propagator Matrices and an Equivariant Linking Number of Knots

In order to compute the Feynman diagrams, one would also need to have a knowledge of the propagators of the bosonic and fermionic fields associated with the kinetic operators L_{boson} and L_{fermion}.

The propagator of the bosonic fields $\triangle_{\text{boson}}$ can be obtained by solving the equation

$$kL_{\text{boson}(IK,\bar{I}\bar{K},ad,\mu\rho)}(x)\times\triangle^{(KJ,\bar{K}\bar{J},db,\rho\nu)}_{\text{boson}}(x-y) = \begin{pmatrix} \delta^I_J & 0 & 0 \\ 0 & \delta^{\bar{I}}_{\bar{J}} & 0 \\ 0 & 0 & \delta^\mu_\nu\delta^a_b \end{pmatrix}\cdot\delta(x-y), \tag{2.63}$$

where L_{boson} is given in (2.31). To first order, the 3×3 matrix $\triangle_{\text{boson}}$ can be written as

$$\triangle_{\text{boson}} \sim \begin{pmatrix} 0 & \triangle^{\varphi\varphi} & \triangle^{\varphi A} \\ \triangle^{\varphi\varphi} & 0 & \triangle^{\varphi A} \\ \triangle^{A\varphi} & \triangle^{A\varphi} & \triangle^{AA} \end{pmatrix}, \tag{2.64}$$

where its components are spanned by all possible boson propagators:

$$\begin{aligned}
\triangle^{(\varphi\varphi)I\bar{J}}(X,G,M;\phi_0,A^\vartheta_0) &= \frac{1}{k}f^{(\varphi\varphi)I\bar{J}}(X,G;\phi_0,A^\vartheta_0)\triangle'^{(\varphi\varphi)}(M;A^\vartheta_0), \\
\triangle^{(AA)ab}_{\mu\nu}(X,G,M;\phi_0,A^\vartheta_0) &= \frac{1}{k_{cs}}f^{(AA)ab}(X,G;\phi_0,A^\vartheta_0)\triangle'^{(AA)}_{\mu\nu}(M;A^\vartheta_0), \\
\triangle^{(\varphi A)I,a}_\mu(X,G,M;\phi_0,A^\vartheta_0) &= \frac{1}{k}f^{(\varphi A)I,a}(X,G;\phi_0,A^\vartheta_0)\triangle'^{(\varphi A)}_\mu(M;A^\vartheta_0).
\end{aligned} \tag{2.65}$$

[4]Here, for convenience, we use "matter fields" to mean ϕ, χ and η.

Here, the labels ϕ_0 and A_0^ϑ mean that the corresponding quantities are evaluated at these values of the covariantly constant map ϕ_0 and flat connection A_0^ϑ. Notice that we can write the propagators as a product of two parts. The first part is a function $f(X, G; \phi_0, A_0^\vartheta)$ on the target manifold X that is characterized by the structural information of X and G. The second part is a function $\triangle^{'}(M; A_0^\vartheta)$ on M.

Similarly, the propagator of the fermionic fields $\triangle_{\text{fermion}}$ can be obtained by solving the equation

$$kL_{\text{fermion}(\bar{I}\bar{K},IK,ad,\mu\rho)}(x)\times\triangle_{\text{fermion}}^{(\bar{K}\bar{J},KJ,db,\rho\nu)}(x-y) = \begin{pmatrix} \delta^{\bar{I}}_{\bar{J}} & 0 & 0 & 0 \\ 0 & \delta^I_J\delta^\mu_\nu & 0 & 0 \\ 0 & 0 & \delta^a_b & 0 \\ 0 & 0 & 0 & \delta^a_b \end{pmatrix}\cdot\delta(x-y), \tag{2.66}$$

where L_{fermion} is given in (2.33). To first order, the 4×4 matrix $\triangle_{\text{fermion}}$ can be written as

$$\triangle_{\text{fermion}} \sim \begin{pmatrix} 0 & \triangle^{\eta\chi} & 0 & 0 \\ \triangle^{\chi\eta} & \triangle^{\chi\chi} & \triangle^{\chi\bar{c}} & 0 \\ 0 & \triangle^{\bar{c}\chi} & 0 & \triangle^{c\bar{c}} \\ 0 & 0 & \triangle^{\bar{c}c} & 0 \end{pmatrix}. \tag{2.67}$$

where its components are spanned by all possible fermion propagators:

$$\begin{aligned}
\triangle_\mu^{(\eta\chi)\bar{I}J}(X,\mathcal{G},M;\phi_0,A_0^\vartheta) &= \frac{1}{k}f^{(\eta\chi)\bar{I}J}(X,\mathcal{G};\phi_0,A_0^\vartheta)\triangle_\mu^{'(\eta\chi)}(M;A_0^\vartheta),\\
\triangle_{\mu\nu}^{(\chi\chi)IJ}(X,\mathcal{G},M;\phi_0,A_0^\vartheta) &= \frac{1}{k}f^{(\chi\chi)IJ}(X,\mathcal{G};\phi_0,A_0^\vartheta)\triangle_{\mu\nu}^{'(\chi\chi)}(M;A_0^\vartheta),\\
\triangle_\mu^{(\chi\bar{c})I,a}(X,\mathcal{G},M;\phi_0,A_0^\vartheta) &= \frac{1}{k}f^{(\chi\bar{c})I,a}(X,\mathcal{G};\phi_0,A_0^\vartheta)\triangle_\mu^{'(\chi\bar{c})}(M;A_0^\vartheta), \quad (2.68)\\
\triangle^{(c\bar{c})ab}(X,\mathcal{G},M;\phi_0,A_0^\vartheta) &= \frac{1}{k}f^{(c\bar{c})ab}(X,\mathcal{G};\phi_0,A_0^\vartheta)\triangle^{'(c\bar{c})}(M;A_0^\vartheta).
\end{aligned}$$

Similar to the boson propagators, we can also write these fermion propagators as the product of two parts.

An Equivariant Linking Number of Knots

Notice here that we may regard $\Delta_{\mu\nu}^{'(\chi\chi)}(M;A_0^\vartheta)$ as an equivariant one-form depending on A_0^ϑ. This means that for one-cycles $C^{'}$ in M which satisfy the following equivariant Stoke's theorem

$$\int_{C'} d_{A_0^\vartheta}\mathcal{F} = \int_{\partial_{A_0^\vartheta}C'} \mathcal{F} = 0, \tag{2.69}$$

the double integral

$$\oint_{C_1'} dx^\mu \oint_{C_2'} dx^\nu \Delta'^{(\chi\chi)}_{\mu\nu}(M; A_0^\vartheta) \tag{2.70}$$

would define an "*equivariant linking number*" of knots C_1' and C_2'.

2.3.6 New Three-Manifold Invariants and Weight Systems

We would now like to show that by computing the perturbative partition function, we would be able to derive *new* three-manifold invariants and their associated weight systems which depend on both $\mathcal{G}$ and X. To this end, let us first review the three-manifold invariants and their associated weight systems that come from Chern-Simons and Rozansky-Witten theory.

Three-Manifold Invariants and Weight Systems From Chern-Simons Theory

The perturbative partition function of Chern-Simons theory can be written as

$$Z_{CS}(M; \mathcal{G}; k_{cs}) = \sum_m Z_{CS}^{(m)}(M; \mathcal{G}; k_{cs}), \tag{2.71}$$

where (m) denotes the order of k_{cs} in the indicated term. If the classical solution A_0 is the trivial flat connection over M, the propagators would be independent of A_0. Then, the partition function would take (up to a one-loop contribution) the very simple form

$$Z_{CS}^{(\mathrm{tr})}(M; \mathcal{G}; k_{cs}) = \exp\left(\sum_{m=1}^{\infty} S_{\mathcal{G},m+1}(M)\, k_{cs}^{-m}\right), \tag{2.72}$$

where

$$S_{\mathcal{G},m+1} = \sum_{\Gamma\in\Gamma_{3,m+1}} a_\Gamma(\mathcal{G}) I_\Gamma(M). \tag{2.73}$$

Here, the sum runs over all trivalent Feynman graphs $\Gamma_{3,m+1}$ with $m+1$ loops (and $2m$ vertices),[5] and $I_\Gamma(M)$ are the integrals over $M \times M \times \cdots \times M$ of the products of propagators.

The Jacobi identity of the Lie algebra of $\mathcal{G}$ is used to show that although the individual integrals $I_\Gamma(M)$ depend on the metric of M, the metric-dependence cancels out of the sum in (2.73) [7, 8]. Thus, $S_{\mathcal{G},m+1}$ and therefore $Z_{CS}(M;\mathcal{G};k_{cs})$, are indeed topological invariants of the three-manifold M. Furthermore, because the factor $a_\Gamma(\mathcal{G})$ can be regarded as a weight factor weighting each graph term, $S_{\mathcal{G},m+1}$ also defines what is called a weight system. Clearly, this weight system depends on Lie algebra structure.

Three-Manifold Invariants and Weight Systems From Rozansky-Witten Theory

The perturbative partition function of Rozansky-Witten theory can (up to a one-loop contribution) be written as

$$Z(M,X) = \sum_\Gamma Z_\Gamma(M,X), \tag{2.74}$$

where $\sum_\Gamma$ is a summation over all relevant Feynman graphs of the theory, and

$$Z_\Gamma(M,X) = b_\Gamma(X)\sum_b I_{\Gamma,b}(M), \tag{2.75}$$

where $\sum_b$ denotes the summation of all possible ways of assigning the vertices to each Feynman graph. Here, $I_{\Gamma,b}$ are the integrals over $M \times M \times \cdots \times M$ of the products of propagators as well as of the relevant one-form fermionic zero modes. $I_{\Gamma,b}$ just depends on the structure of M, while b_Γ serves as a weight factor which depends on the curvature tensor of the target space X that comes from the underlying vertices. Thus, $Z_\Gamma(M,X)$ defines a weight system. Clearly, this weight system depends on hyperkähler geometry.

The Bianchi identity plays the same role here as the Jacobi identity in CS theory [10]; one can use it to show that the dependence on the metric of M cancels out of the sum (2.74), i.e., $Z(M,X)$ is a topological invariant of the three-manifold M.

[5] For a description of a (trivalent) Feynman graph, see [10].

Coming Back to Our Theory

Coming back to our theory, we can, after evaluating the path integral, write the perturbative partition function as

$$Z(M, X, \mathcal{G}) = \sum_{A_0^\vartheta} e^{-\int_M k_{cs} L_{cs}(A_0^\vartheta)} \cdot Z_0(A_0^\vartheta) \cdot Z(M, X, \mathcal{G}; A_0^\vartheta; k_{cs}), \tag{2.76}$$

where $e^{-\int_M k_{cs} L_{cs}(A_0^\vartheta)}$ is the topological factor coming from the Chern-Simons part of the total Lagrangian evaluated at a flat connection A_0^ϑ; $Z_0(A_0^\vartheta)$ is the topological one-loop contribution given in (2.46); and

$$Z(M, X, \mathcal{G}; A_0^\vartheta; k_{cs}) = \sum_\Gamma Z_\Gamma(M, X, \mathcal{G}; A_0^\vartheta; k_{cs}^m), \tag{2.77}$$

where $\sum_\Gamma$ is a sum over all possible Feynman diagrams with two or more loops that (i) have the right number of fermionic zero modes to absorb those that appear in the path integral measure, and (ii) are free of the coupling constant k. Here, the label k_{cs}^m (where m may vanish) means that Γ carries with it a factor of k_{cs}^m.

In fact, Z_Γ can be expressed as

$$Z_\Gamma(M, X, \mathcal{G}; A_0^\vartheta; k_{cs}^m) = \int_{\mathcal{M}^\vartheta} \sqrt{g}\, d^{2n}\phi_0^I\, d^{2n}\phi_0^{\bar{I}}\, W_\Gamma(X, \mathcal{G}; \phi_0, A_0^\varphi) I_\Gamma(M, X, \mathcal{G}; \phi_0, A_0^\vartheta; k_{cs}^m), \tag{2.78}$$

where I_Γ is an integral over $M \times M \times \cdots \times M$ of the products of propagators as well as of the one-form fermionic zero modes $\omega_\mu(x)$ in (2.50), while the weight factor W_Γ is a product of terms relevant to Γ that are associated with the vertices in Fig. 1.

We can characterize the partition function by classifying the Feynman diagrams into three categories as follows.

(1) Chern-Simons-Type Diagrams. These diagrams result purely from the vertices $A \wedge A \wedge A$. Thus, they correspond to diagrams in usual Chern-Simons theory. The topological property of Chern-Simons-type diagrams has already been verified in earlier works [7, 8]. As such, we would have nothing more to add

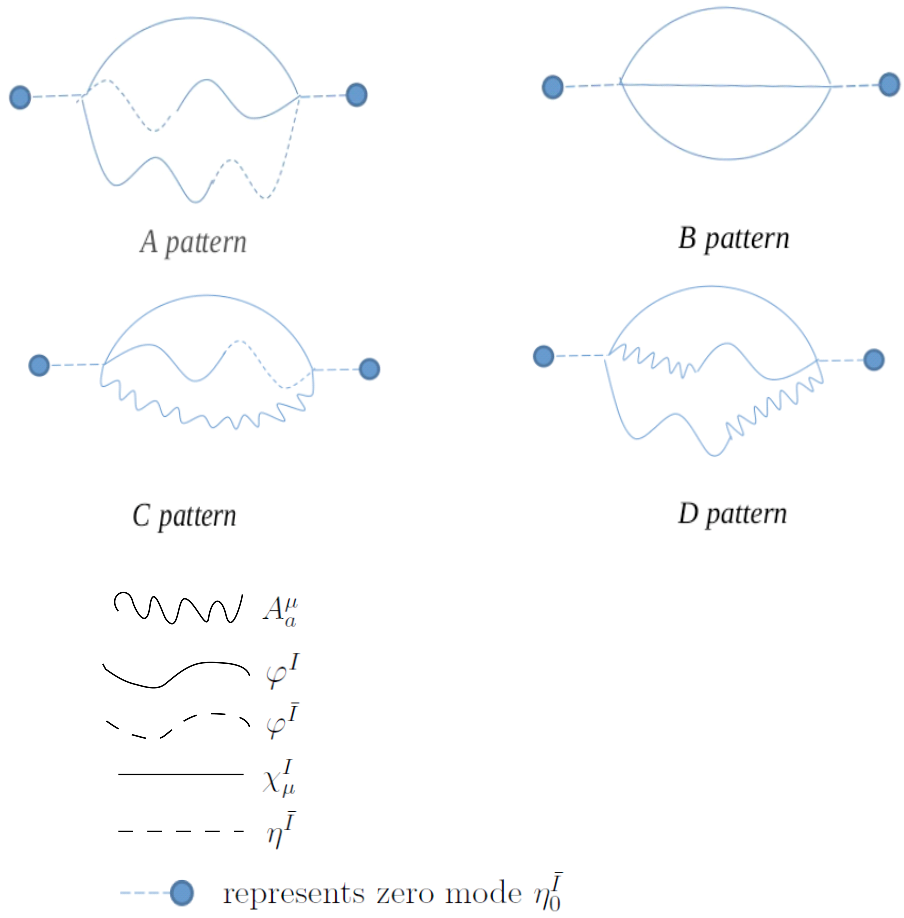

FIGURE 2.2: A, B, C and D Pattern Diagrams

about them.

To discuss the next two types of diagrams, we take, for simplicity, the case where $b_1' = 0$ and $b_0' = 1$. Then, the nonvanishing Feynman diagrams must contain exactly $2nb_0'$ zero modes η_0^I. For brevity, we will only discuss diagrams whose vertices emanate 4 legs.

(2) <u>Diagrams Free of Gauge Fields</u>. Since these diagrams result from vertices which are free of the gauge field A, they do not contain the gauge field propagator. Examples of such diagrams are given by the A pattern and B pattern in Fig. 2.

The A pattern diagram is formed by the vertex

$$k\partial_K\Gamma_{I\bar{M}\bar{N}}\partial_\mu\varphi^{\bar{M}}\varphi^K\chi^{I,\mu}\eta_0^{\bar{N}},$$

and the propagators in the diagram are $\triangle^{(\chi\chi)}$ and $\triangle^{(\varphi\varphi)}$. Therefore for the A pattern diagram, the terms in (2.78) are

$$\boxed{I_\Gamma(M,X,\mathcal{G};\phi_0,A_0^\vartheta)=\int_M\prod_{l=1}^{n}(\triangle_{\mu\nu}^{(\chi\chi)I_lJ_l}\partial^\mu\triangle^{(\varphi\varphi)L_l\bar{P}_l}\partial^\nu\triangle^{(\varphi\varphi)K_l\bar{Q}_l})(x_l,y_l)\,d^3x_ld^3y_l} \tag{2.79}$$

and

$$\boxed{W_\Gamma(X,\mathcal{G};\phi_0^I)=\epsilon^{\bar{M}_1\ldots\bar{M}_{2n}}\prod_{l=1}^{n}(\partial_{K_l}\partial_{\bar{M}_{1,l}}g_{\bar{P}_lI_l})(\partial_{L_l}\partial_{\bar{M}_{2,l}}g_{\bar{Q}_lJ_l})} \tag{2.80}$$

where the $\epsilon^{\bar{M}_1\ldots\bar{M}_{2n}}$ factor comes from the expectation value of the zero modes η_0 defined in (2.49). The contribution of this diagram to the partition function can then be evaluated by substituting the above two expressions in (2.78).

The B pattern diagram is formed by the vertex

$$k\Omega_{IJ}R^J_{KL\bar{M}}\epsilon^{\mu\nu\rho}\chi^I_\mu\chi^K_\nu\chi^L_\rho\eta_0^{\bar{M}},$$

and the propagator in the diagram is $\triangle^{(\chi\chi)}$. Therefore for the B pattern diagram, the terms in (2.78) are

$$\boxed{I_\Gamma(M,\mathcal{G},X;\phi_0^I,A_0^\vartheta)=\int_M\prod_{l=1}^{n}(\epsilon^{\mu_1\nu_1\rho_1}\epsilon^{\mu_2\nu_2\rho_2}\triangle_{\mu_1\mu_2}^{(\chi\chi)I_{1,l}I_{2,l}}\triangle_{\nu_1\nu_2}^{(\chi\chi)K_{1,l}K_{2,l}}\triangle_{\rho_1\rho_2}^{(\chi\chi)L_{1,l}L_{2,l}})(x_l,y_l)\,d^3x_ld^3y_l} \tag{2.81}$$

and

$$\boxed{W_\Gamma(X,\mathcal{G};\varphi_0^I)=\epsilon^{\bar{M}_1\ldots\bar{M}_{2n}}\prod_{l=1}^{n}(\Omega_{I_{1,l}J_{1,l}}R^{J_{1,l}}_{K_{1,l}L_{1,l}\bar{M}_{1,l}})(\Omega_{I_{2,l}J_{2,l}}R^{J_{2,l}}_{K_{2,l}L_{2,l}\bar{M}_{2,l}})} \tag{2.82}$$

Notice that the A and B pattern diagrams in Fig. 2 are similar to those in RW theory. Nevertheless, unlike RW theory, our propagator factor I_Γ depends on the flat gauge field A_0^ϑ. If A_0^ϑ were trivial, the contributions of the A and B pattern diagrams to the partition function would be as given in RW theory, as expected.

(3) Diagrams Mixing Gauge and Matter Fields. Examples of such diagrams are given by the C pattern and D pattern in Fig. 2.

The C pattern diagram is formed by the vertices

$$k\partial_K(\nabla_{\bar{M}}V_I^a)\eta_0^{\bar{M}}\chi_\mu^I A_a^\mu \varphi^K \quad \text{and} \quad k\partial_{\bar{K}}(\nabla_{\bar{M}}V_I^a)\eta_0^{\bar{M}}\chi_\mu^I A_a^\mu \varphi^{\bar{K}},$$

and the propagators in the diagram are $\triangle^{(\chi\chi)}$, $\triangle^{(\varphi\varphi)}$, $\triangle^{(AA)}$. Therefore for the C pattern diagram, the terms in (2.78) are

$$\boxed{I_\Gamma(M,X,\mathcal{G};\phi_0^I,A_0^\vartheta;k_{cs}^{-n}) = \int_M \prod_{l=1}^n (\triangle_{\mu\nu}^{(\chi\chi)I_lJ_l}\triangle^{(\varphi\varphi)K_l\bar{L}_l}\triangle_{a_lb_l}^{(AA)\mu\nu})(x_l,y_l)\, d^3x_l d^3y_l} \tag{2.83}$$

and

$$\boxed{W_\Gamma(X,\mathcal{G};\varphi_0^I) = \epsilon^{\bar{M}_1\ldots\bar{M}_{2n}}\prod_{l=1}^n \partial_{\bar{L}_l}(\nabla_{\bar{M}_{1,l}}V_{I_l}^{a_l})\partial_{K_l}(\nabla_{\bar{M}_{2,l}}V_{J_l}^{b_l})} \tag{2.84}$$

The D pattern diagram is formed by the vertex

$$k\partial_K(\nabla_{\bar{M}}V_I^a)\eta_0^{\bar{M}}\chi_\mu^I A_a^\mu \varphi^K,$$

and the propagators in the diagram are $\triangle^{(\chi\chi)}$, $\triangle^{(\varphi A)}$. Therefore for the D pattern diagram, the terms in (2.78) are

$$\boxed{I_\Gamma(M,X,\mathcal{G};\phi_0^I,A_0^\vartheta) = \int_M \prod_{l=1}^n (\triangle_{\mu a_l}^{(\varphi A)K_{1,l}}\triangle_{\nu b_l}^{(\varphi A)K_{2,l}}\triangle^{(\chi\chi)I_{1,l}I_{2,l},\mu\nu})(x_l,y_l)\, d^3x_l d^3y_l} \tag{2.85}$$

and

$$\boxed{W_\Gamma(X,\mathcal{G};\varphi_0^I) = \epsilon^{\bar{M}_1\ldots\bar{M}_{2n}}\prod_{l=1}^n \partial_{K_{1,l}}(\nabla_{\bar{M}_{1,l}}V_{I_{1,l}}^{a_l})\partial_{K_{2,l}}(\nabla_{\bar{M}_{2,l}}V_{I_{2,l}}^{b_l})} \tag{2.86}$$

As shown in (2.65) and (2.68), the propagators $\triangle$ can be expressed as the product of a function $f(X,G)$ on the target manifold X and a function $\triangle^{'}(M)$ on the three-manifold M. Therefore, we can rewrite the above propagator factors as

$$\boxed{I_\Gamma(M,X,\mathcal{G};\phi_0,A_0^\vartheta;k_{cs}^m) = f_\Gamma(X,\mathcal{G};\phi_0,A_0^\vartheta)I_\Gamma^{'}(M;A_0^\vartheta;k_{cs}^m)} \tag{2.87}$$

where $I_\Gamma^{'}$ is a function on M that depends on the flat gauge field A_0^ϑ and which carries a factor of k_{cs}^m, and the function f_Γ is characterized, among other things, by the structure of the target space X and the gauge group $\mathcal{G}$. In turn, this means that we can rewrite (2.78) as

$$Z_\Gamma(M,X,\mathcal{G};A_0^\vartheta;k_{cs}^m) = \mathscr{W}_\Gamma(X,\mathcal{G};A_0^\vartheta)\, I_\Gamma^{'}(M;A_0^\vartheta;k_{cs}^m) \tag{2.88}$$

where

$$\boxed{\mathscr{W}_\Gamma(X,\mathcal{G};A_0^\vartheta) = \int_{\mathcal{M}^\vartheta} \sqrt{g}\, d^{2n}\phi_0^I\, d^{2n}\phi_0^{\bar{I}}\; W_\Gamma(X,\mathcal{G};\phi_0,A_0^\vartheta) f_\Gamma(X,\mathcal{G};\phi_0,A_0^\vartheta)} \tag{2.89}$$

can be regarded as a weight factor which combines the structural information of the hyperkähler manifold X and the Lie algebra $\mathfrak{g}$ of the gauge group $\mathcal{G}$.

As in CS and RW theory, $I_\Gamma^{'}$ in (2.88) can be expected to depend on the metric of M. However, since the partition function in (2.76) and therefore

$$\boxed{Z(M,X,\mathcal{G};A_0^\vartheta;k_{cs}) = \sum_\Gamma \mathscr{W}_\Gamma(X,\mathcal{G};A_0^\vartheta)\, I_\Gamma^{'}(M;A_0^\vartheta;k_{cs}^m)} \tag{2.90}$$

are topological on M at the outset, the metric-dependence of $I_\Gamma^{'}$ should cancel out in the sum (2.90). To rigorously show this cancellation, we can use the Jacobi identity of $\mathcal{G}$, the Bianchi identity of X, and the geometric identities of the moment maps discussed in section 2.1. However, at each order of k_{cs}, the partition function and consequently, its variation with respect to the metric of M, contains so many different terms that it would be a formidable task to demonstrate this cancellation using our purely physical methods. We hope that in the near future, novel and sophisticated methods would be devised to facilitate this explicit verification.

In summary, our perturbative partition function furnishes us with a *new* three-manifold invariant $Z(M, X, \mathcal{G}; A_0^\vartheta; k_{cs})$ which depends on both $\mathcal{G}$ and X, that also defines a *new* weight system whose weights $\mathscr{W}_\Gamma(X, \mathcal{G}; A_0^\vartheta)$ are characterized by both Lie algebra structure *and* hyperkähler geometry.

2.4 Canonical Quantization and the Nonperturbative Partition Function

Let us now canonically quantize our gauged sigma model on M with target space X. To this end, first note that the Hamiltonian $H = \langle \int T_{00} \rangle = \langle \delta_{\hat{Q}} \mathcal{O} \rangle = 0$. Next, note that locally, the Riemannian manifold M can be written as $\Sigma \times I$, where I is the 'time' dimension and Σ is a compact Riemann surface. Thus, since $H = 0$ whence the theory should be time-independent, it would mean that we can just analyze the physics over any $\Sigma \times I \subset M$.

This property of $H = 0$ also means that only ground states contribute to the spectrum of the theory. Therefore, where the fermions are concerned, only the zero modes contribute to the physical Hilbert space. Where the gauge field is concerned, only the classical configuration of flat connections A_0 contribute to the physical Hilbert space. And where the bosons are concerned, only the covariantly constant maps ϕ from M to X which satisfy $D_\mu \phi = \partial_\mu \phi + A^a_{0\mu} V_a = 0$, contribute to the physical Hilbert space.

Let τ be the time coordinate. Then, according to the last paragraph, ϕ would satisfy $\partial_\tau \phi + A^a_{0\tau} V_a = 0$. In the gauge where $A_\tau = 0$, we would also have

$$\partial_\tau \eta^{\bar{I}} = 0, \quad \partial_\tau \chi^I_\tau = 0, \quad \partial_\tau \chi^I_\mu = 0, \tag{2.91}$$

where $\eta^{\bar{I}}$, χ^I_τ and χ^I_μ are fermionic zero modes. In other words, the zero modes ϕ, η^I, χ^I_τ and χ^I_μ are τ-independent, which means that we effectively have a two-dimensional gauged sigma model on Σ.

The Commutation and Anticommutation Relations

From the Lagrangian, we compute the momentum conjugate of η, χ and A to be

$$\begin{aligned}
\frac{\delta L}{\delta \partial^\tau \eta^{\bar{K}}} &= g_{I\bar{K}} \chi^I_\tau, \\
\frac{\delta L}{\delta \partial_\tau \chi^I_\mu} &= \epsilon^{\tau\mu\nu} \Omega_{IJ} \chi^J_\nu, \\
\frac{\delta L}{\delta \partial^\tau \chi^I_\tau} &= g_{\bar{K}I} \eta^{\bar{K}}, \\
\frac{\delta L}{\delta \partial_\tau A^a_{0\mu}} &= \epsilon^{\tau\mu\nu} k_{ab} A^b_{0\nu}.
\end{aligned} \tag{2.92}$$

Note at this point from (2.33) that the fermionic zero modes χ^I_μ are solutions of the covariant equation

$$\Omega_{IJ} \epsilon^{\tau\mu\nu} \partial_\mu \chi^J_\nu + \Omega_{IK} \epsilon^{\tau\mu\nu} \partial_J V^K_a A^a_{0\mu} \chi^J_\nu = 0; \quad \mu, \nu = 1, 2, \tag{2.93}$$

which depend on the choice of the flat connection A_0; in other words, we can write $\chi^I_\mu(x) = \chi^I_\alpha \omega^\alpha_\mu(A_0, x)$, where ω^α are covariant harmonic one-forms on Σ, and χ^I_α are constant fermionic coefficients. Hence, if $\int_\Sigma \omega^\alpha \wedge \omega^\beta = L^{\alpha,\beta}$, the relations in (2.92) tell us that the commutation and anticommutation relations upon quantizing the zero modes must be

$$\begin{aligned}
\{\eta^{\bar{I}}, \chi^J_\tau\} &= \frac{1}{k} g^{\bar{I}J}, \\
\{\chi^I_\alpha, \chi^J_\beta\} &= \frac{1}{k} \Omega^{IJ} (L^{-1})_{\alpha,\beta}, \\
\left[A^a_{0\mu}(x) A^b_{0\nu}(y)\right] &= \frac{1}{k_{cs}} \epsilon_{\mu\nu} \delta^{ab} \delta^2(x-y),
\end{aligned} \tag{2.94}$$

where g and Ω are evaluated at the covariantly constant map ϕ.

A Relevant Digression

Before proceeding any further, let us discuss the following important point. Recall from section 3 that after gauge-fixing, it is the $\hat{Q}$-cohomology that is relevant. Nevertheless, modulo gauge transformations the spectrum of the theory is unchanged by gauge-fixing, and so the Q- and $\hat{Q}$-cohomology ought to be equivalent. Let us now verify this claim.

First, recall that we have

$$Q^2 = \text{gauge transformation}, \tag{2.95}$$

and

$$\hat{Q} = Q + Q_{\mathrm{FP}}, \quad \text{where} \quad \hat{Q}^2 = 0. \tag{2.96}$$

Second, by definition, we have

$$\ker(\hat{Q}) = \{\mathcal{O} | \{Q + Q_{\mathrm{FP}}, \mathcal{O}] = 0\}. \tag{2.97}$$

From (2.11), we find that $\{Q, \mathcal{O}] \neq -\{Q_{\mathrm{FP}}, \mathcal{O}]$. So,

$$\ker(\hat{Q}) = \ker(Q) \cap \ker(Q_{\mathrm{FP}}). \tag{2.98}$$

That is, $\{Q, \mathcal{O}] = 0$ and $\{Q_{\mathrm{FP}}, \mathcal{O}] = 0$, which means

$$\{\{Q, Q_{\mathrm{FP}}\}, \mathcal{O}] = 0. \tag{2.99}$$

Third, from the definition of Q_{FP}, we have

$$\{Q_{\mathrm{FP}}^2, \mathcal{O}] = 0. \tag{2.100}$$

Also, we have

$$\{\hat{Q}^2, \mathcal{O}] = \{Q^2 + \{Q, Q_{\mathrm{FP}}\} + Q_{\mathrm{FP}}^2, \mathcal{O}] = 0. \tag{2.101}$$

Thus, from (2.99), (2.100) and (2.101), we have

$$\{Q^2, \mathcal{O}] = 0. \tag{2.102}$$

In turn, this means that

$$\ker(\hat{Q}) = \ker(Q) \cap \ker(Q_{\mathrm{FP}}) = \ker(Q) \cap \{\mathcal{O} | \{Q^2, \mathcal{O}] = 0\}. \tag{2.103}$$

Last, note that in

$$\mathrm{im}(\hat{Q}) = \mathrm{im}(Q + Q_{\mathrm{FP}}), \tag{2.104}$$

because $\mathrm{im}(Q_{\mathrm{FP}})$ contains ghost fields, $\mathrm{im}(Q_{\mathrm{FP}})$ does not contribute to the physical Hilbert space. Hence,

$$\mathrm{im}(\hat{Q}) \cong \mathrm{im}(Q). \tag{2.105}$$

Altogether, this means that

$$\mathcal{O} \in \frac{\ker(\hat{Q})}{\mathrm{im}(\hat{Q})} = \frac{\ker(Q) \cap \ker(Q_{\mathrm{FP}})}{\mathrm{im}(Q + Q_{\mathrm{FP}})} \cong \frac{\ker(Q) \cap \{\mathcal{O} | \{Q^2, \mathcal{O}] = 0\}}{\mathrm{im}(Q)}, \tag{2.106}$$

which verifies our claim that the $\hat{Q}$- and Q-cohomology are equivalent, modulo gauge transformations. Therefore, let us henceforth focus on the Q-cohomology; in particular, let us proceed to ascertain the relevant Hilbert space of states in the Q-cohomology, where the Hilbert space is $\mathcal{G}$-orbit space.

The Hilbert Space of States

To this end, note that since we are restricting ourselves to the classical configuration A_0 that is free of interacting fluctuations, we can view the total theory as a CS theory plus a non-dynamically gauged RW theory. As such, any state $|\Psi\rangle$ in the Q-cohomology ought to take the form

$$\boxed{|\Psi\rangle = |\psi\rangle \otimes \tilde{\Phi}|0\rangle} \tag{2.107}$$

Here, $|\psi\rangle$ is a state in the CS theory which is associated with a Q-closed but not Q-exact wave function $\psi(A_0^\vartheta)$ that depends on a flat gauge field A_0^ϑ along Σ, where [46]

$$\psi(A_0^\vartheta) = \int_{A|_\Sigma = A_0^\vartheta} DA\, e^{-S_{cs}}, \tag{2.108}$$

and $\tilde{\Phi}$ is a Q-closed but not Q-exact state operator of the non-dynamically gauged RW theory. Let us now determine $\tilde{\Phi}$.

From (2.94), it is clear that the vacuum state $|0\rangle$ would be annihilated by the operators χ_β^I and χ_τ^I.[6] Hence, a first-cut construction of an arbitrary state $|\Phi\rangle$ of the non-dynamically gauged RW theory would be

$$|\Phi\rangle = \tilde{\Phi}|0\rangle = \Phi_{I_1 \cdots I_l \bar{I}_1 \cdots \bar{I}_k}(\phi) \chi_\alpha^{I_1} \cdots \chi_\alpha^{I_l} \eta^{\bar{I}_1} \cdots \eta^{\bar{I}_k} |0\rangle. \tag{2.109}$$

[6]For ease of illustration, we henceforth assume that $b_0^{'} = 1$ and $b_1^{'} = 1$.

Generically, Φ depends on the covariantly constant map ϕ; hence, a natural generically-nonvanishing scalar product of states would be given by

$$\langle\Phi^{(1)}|\Phi^{(2)}\rangle = \int_{\mathcal{M}^\vartheta} \sqrt{g}\, d^{2n}\phi^I d^{2n}\phi^{\bar{I}}\, \epsilon^{I_1\dots I_{2n}\bar{I}_1\dots\bar{I}_{2n}}\, \Phi^{(1)}_{I_1\cdots I_q\bar{I}_1\cdots\bar{I}_p} \Phi^{(2)}_{I_{q+1}\cdots I_{2n}\bar{I}_{p+1}\cdots\bar{I}_{2n}} = \int_{\mathcal{M}^\vartheta} \Phi^{q+p}\wedge\Phi^{4n-q-p}, \tag{2.110}$$

where Φ^m is an m-form on $\mathcal{M}^\vartheta$, the space of all physically distinct ϕ's for some A_0^ϑ. In other words, $\tilde{\Phi}$ would correspond to an element of $\Omega^{l+k}(\mathcal{M}^\vartheta)$, the space of all $(l+k)$-forms on $\mathcal{M}^\vartheta$.

Now, from (2.8), the fields transform under the supercharge Q as

$$\begin{aligned} \delta_Q\eta^{\bar{I}_i} &= -V_a^{\bar{I}_i}\mu_+^a, \\ \delta_Q\phi^{\bar{I}} &= \eta^{\bar{I}}, \\ \delta_Q\chi^I &= D\phi^I. \end{aligned} \tag{2.111}$$

At the level of zero modes, the last equation $\delta_Q\chi^I = D\phi^I = 0$. If $\tilde{\Phi}$ is d-closed, i.e., $\partial_{\bar{J}}\Phi(\phi) = \partial_J\Phi(\phi) = 0$, we would have

$$\begin{aligned} \delta_Q\tilde{\Phi} &= \sum_i (-1)^{l+i}\Phi_{I_1\cdots I_l\bar{I}_1\cdots\bar{I}_k} V_a^{\bar{I}_i}\mu_+^a\chi_\alpha^{I_1}\cdots\chi_\alpha^{I_l}\eta^{\bar{I}_1}\cdots\hat{\eta}^{\bar{I}_i}\cdots\eta^{\bar{I}_k} \\ &= (-1)^l\, k\,\Phi_{I_1\cdots I_l\bar{I}_1\bar{I}_2\cdots\bar{I}_k} V_a^{\bar{I}_1}\mu_+^a\chi_\alpha^{I_1}\cdots\chi_\alpha^{I_l}\eta^{\bar{I}_2}\cdots\eta^{\bar{I}_k} \\ &\neq 0. \end{aligned} \tag{2.112}$$

Thus, the state operator $\tilde{\Phi}$ is not Q-closed, as we would like it to be.

We can try to 'improve' it to

$$\tilde{\Phi} = \Phi_{I_1\cdots I_l\bar{I}_1\cdots\bar{I}_k}(\phi)\chi_\alpha^{I_1}\cdots\chi_\alpha^{I_l}\eta^{\bar{I}_1}\cdots\eta^{\bar{I}_k} - \mu_+^a\Phi_{aI_1\cdots I_l\bar{I}_1\cdots\bar{I}_{k-2}}(\phi)\chi_\alpha^{I_1}\cdots\chi_\alpha^{I_l}\eta^{\bar{I}_1}\cdots\eta^{\bar{I}_{k-2}}, \tag{2.113}$$

where now,

$$\begin{aligned} \delta_Q\tilde{\Phi} =& (-1)^l\,[k\,V_a^{\bar{I}_1}\Phi_{I_1\cdots I_l\bar{I}_1\bar{I}_2\cdots\bar{I}_k}]\,\mu_+^a\chi_\alpha^{I_1}\cdots\chi_\alpha^{I_l}\eta^{\bar{I}_2}\cdots\eta^{\bar{I}_k} \\ &- (-1)^l\,[\partial_{\bar{K}}\Phi_{aI_1\cdots I_l\bar{I}_1\cdots\bar{I}_{k-2}}]\,\mu_+^a\chi_\alpha^{I_1}\cdots\chi_\alpha^{I_l}\eta^{\bar{K}}\eta^{\bar{I}_1}\cdots\eta^{\bar{I}_{k-2}} \\ &- (-1)^l(k-2)\mu_+^a\mu_+^b V_b^{\bar{I}_1}\Phi_{aI_1\cdots I_l\bar{I}_1\cdots\bar{I}_{k-2}}\chi_\alpha^{I_1}\cdots\chi_\alpha^{I_l}\eta^{\bar{I}_2}\cdots\eta^{\bar{I}_{k-2}}, \end{aligned} \tag{2.114}$$

after exploiting the fact that μ_+ is holomorphic. If moreover, $\Phi_{aI_1\cdots I_l\bar{I}_1\cdots\bar{I}_{k-2}}$ is anti-holomorphic and

$$\iota_a(\Phi) = d\Phi_a, \tag{2.115}$$

where $(\iota_a(\Phi))_{\bar{I}_2\cdots I_l\bar{I}_1\bar{I}_2\cdots\bar{I}_k} = k\, V_a^{\bar{I}_1}\Phi_{I_1\cdots I_l\bar{I}_1\bar{I}_2\cdots\bar{I}_k}$ is a contraction with V_a of $\Phi \in \Omega^{l+k}(\mathcal{M}^\vartheta)$, and $(d\Phi_a)_{\bar{K}I_1\cdots I_l\bar{I}_1\cdots\bar{I}_{k-2}} = \partial_{\bar{K}}\Phi_{aI_1\cdots I_l\bar{I}_1\cdots\bar{I}_{k-2}}$ is an exterior derivative of $\Phi_a \in \mathfrak{g}^* \otimes \Omega^{l+k-2}(\mathcal{M}^\vartheta)$, the second term on the RHS of (2.114) would simply cancel the first one out, i.e., we would have

$$\delta_Q\tilde{\Phi} = (-1)^{l+1}(k-2)\mu_+^a\mu_+^b V_b^{\bar{I}_1}\Phi_{aI_1\cdots I_l\bar{I}_1\cdots\bar{I}_{k-2}}\chi_\alpha^{I_1}\cdots\chi_\alpha^{I_l}\eta^{\bar{I}_2}\cdots\eta^{\bar{I}_{k-2}}. \tag{2.116}$$

Hence, by 'improving' $\tilde{\Phi}$ via (2.113), we have actually made progress: by comparing (2.116) and (2.112), it is clear that we have gone from having k to $k-2$ many η fields in the expression for $\delta_Q\tilde{\Phi}$.

We can continue to 'improve' $\tilde{\Phi}$ by adding more terms of lower order in η:

$$\begin{aligned}\tilde{\Phi} =& \Phi_{I_1\cdots I_l\bar{I}_1\cdots\bar{I}_k}(\phi)\chi_\alpha^{I_1}\cdots\chi_\alpha^{I_l}\eta^{\bar{I}_1}\cdots\eta^{\bar{I}_k} - \mu_+^a\Phi_{aI_1\cdots I_l\bar{I}_1\cdots\bar{I}_{k-2}}(\phi)\chi_\alpha^{I_1}\cdots\chi_\alpha^{I_l}\eta^{\bar{I}_1}\cdots\eta^{\bar{I}_{k-2}}\\ &+ \mu_+^a\mu_+^b\Phi_{abI_1\cdots I_l\bar{I}_1\cdots\bar{I}_{k-4}}(\phi)\chi_\alpha^{I_1}\cdots\chi_\alpha^{I_l}\eta^{\bar{I}_1}\cdots\eta^{\bar{I}_{k-4}}\\ &- \mu_+^a\mu_+^b\mu_+^c\Phi_{abcI_1\cdots I_l\bar{I}_1\cdots\bar{I}_{k-6}}(\phi)\chi_\alpha^{I_1}\cdots\chi_\alpha^{I_l}\eta^{\bar{I}_1}\cdots\eta^{\bar{I}_{k-6}} + \dots.\end{aligned} \tag{2.117}$$

Here, even-valued k is such that $0 < k \leq \dim_{\mathbb{C}}(\mathcal{M}^\vartheta)$, and $\Phi_a, \Phi_{ab}, \Phi_{abc}, \dots \in S(\mathfrak{g}^*) \otimes \Omega(\mathcal{M}^\vartheta)$ are anti-holomorphic, where $S(\mathfrak{g}^*)$ is the symmetric algebra on $\mathfrak{g}^*$. If moreover,

$$\iota_a(\Phi) = \mathrm{d}\Phi_a, \quad \iota_b(\Phi_a) = \mathrm{d}\Phi_{ab}, \quad \iota_c(\Phi_{ab}) = \mathrm{d}\Phi_{abc}, \dots, \tag{2.118}$$

one will find that $\delta_Q\tilde{\Phi} = 0$.

From the field variations in (2.111) and the comment thereafter, one can see that Q effectively acts on $\tilde{\Phi}$ as $d - \mu_+^a\iota_a$. Together with (2.117) and (2.118), it would mean that for $\tilde{\Phi}$ to be Q-closed but not Q-exact, it must correspond to a class in the G-equivariant cohomology $H_G(\mathcal{M}^\vartheta)$.

It is now clear from (2.107) and the fact that $\tilde{\Phi}$ corresponds to a class in $H_G(\mathcal{M}^\vartheta)$, that the relevant Hilbert space $\mathcal{H}$ of all states $|\Psi\rangle$ in the Q-cohomology can be expressed as

$$\boxed{\mathcal{H} = \mathcal{H}_{\text{CS}}(A_0^\vartheta, \Sigma) \otimes H_G(\mathcal{M}^\vartheta)} \tag{2.119}$$

where $\mathcal{H}_{\text{CS}}(A_0^\vartheta, \Sigma)$ is the Hilbert space of states in CS theory associated with wave functions $\psi(A_0^\vartheta)$ in the Q-cohomology that depend on a flat gauge field A_0^ϑ on Σ.

An Example

Before we end this subsection, let us consider the case where $\Sigma = \mathbf{S}^2$, G is some arbitrary compact simple Lie group, $X = T^*(G/\mathbb{T})$, and $\mathbb{T} \subset G$ is a maximal torus. For simply-connected $\Sigma = \mathbf{S}^2$, we can go to pure gauge on Σ whence we can regard the flat gauge field A_0^ϑ to be trivial in all directions (since $A_{0\tau}^\vartheta = 0$ also). Consequently, $\mathcal{H}_{\text{CS}}$ is trivial, χ_μ and η would become ordinary harmonic forms on Σ, and the ϕ's would just be constant maps whence $\mathcal{M}^\vartheta = X = T^*(G/\mathbb{T})$. Therefore, the corresponding Hilbert space would simply be

$$\mathcal{H}_G = H_G(T^*(G/\mathbb{T})), \tag{2.120}$$

which is the G-equivariant cohomology of $T^*(G/\mathbb{T})$. In other words, via the Cartan model of equivariant cohomology, we have (c.f. [47])

$$\mathcal{H}_G \cong H\left([S(\mathfrak{g}^*) \otimes \Omega(T^*(G/\mathbb{T}))]_{G\text{-invariant}}\right). \tag{2.121}$$

Here, $H(\dots)$ is the cohomology of the complex with Cartan differential $d_G = 1 \otimes d + F^a \otimes \iota_a$, where F^a is some $\mathfrak{g}$-valued function on $T^*(G/\mathbb{T})$ of degree two.

From our discussion leading up to (2.117), and the fact that $b_1(\mathbf{S}^2) = 0$ and $b_0(\mathbf{S}^2) = 1$ whence there are no χ_μ's but $\dim_{\mathbb{C}}(T^*(G/\mathbb{T}))$ many η's, we find that a generic arbitrary state in $\mathcal{H}_G$ would be given by

$$|\Psi\rangle = \tilde{\Phi}|0\rangle, \tag{2.122}$$

where

$$\begin{aligned}\tilde{\Phi} =&\Phi_{\bar{I}_1\cdots\bar{I}_k}(\phi)\eta^{\bar{I}_1}\cdots\eta^{\bar{I}_k} - \mu_+^a\Phi_{a\bar{I}_1\cdots\bar{I}_{k-2}}(\phi)\eta^{\bar{I}_1}\cdots\eta^{\bar{I}_{k-2}}\\ &+\mu_+^a\mu_+^b\Phi_{ab\bar{I}_1\cdots\bar{I}_{k-4}}(\phi)\eta^{\bar{I}_1}\cdots\eta^{\bar{I}_{k-4}}\\ &-\mu_+^a\mu_+^b\mu_+^c\Phi_{abc\bar{I}_1\cdots\bar{I}_{k-6}}(\phi)\eta^{\bar{I}_1}\cdots\eta^{\bar{I}_{k-6}}+\ldots.\end{aligned} \tag{2.123}$$

Here, even-valued k is such that $0 < k \leq \dim_{\mathbb{C}}(T^*(G/\mathbb{T}))$, and $\Phi_a, \Phi_{ab}, \Phi_{abc}, \cdots \in S(\mathfrak{g}^*)\otimes\Omega(T^*(G/\mathbb{T}))$ are anti-holomorphic.

That being said, it can be shown [48] that $\mathcal{H}_G = S(\mathfrak{t}^*)$, where $\mathfrak{t}$ is the Lie algebra of T. In other words, an arbitrary state in $\mathcal{H}_G$ ought to be given by

$$\boxed{|\Psi\rangle = (-1)^p\,\mu_+^{a_1}\mu_+^{a_2}\cdots\mu_+^{a_p}\Phi_{a_1a_2\ldots a_p}(\phi)|0\rangle} \tag{2.124}$$

where $1 \leq a_i \leq \mathrm{rank}(G)$, and p is any positive integer.

Take for example $G = SU(2)$ and $X = T^*(\mathbf{CP}^1)$, where $\mathrm{rank}(G) = 1$ and $\dim_{\mathbb{C}}(X) = 2$. Then, the only state in $\mathcal{H}_{SU(2)}$ is

$$|\Psi^{(1)}\rangle \;=\; -\mu_+^1\Phi_1(\phi)|0\rangle. \tag{2.125}$$

Take as another example $G = SU(N)$ and $X = T^*(SU(N)/U(1)^{N-1})$, where $\mathrm{rank}(G) = N-1$ and $\dim_{\mathbb{C}}(X) = N(N-1)$. Then, the independent states in $\mathcal{H}_{SU(N)}$ ought to take the form

$$|\Psi^{(i)}\rangle = (-1)^i\mu_+^{a_1}\mu_+^{a_2}\cdots\mu_+^{a_i}\Phi_{a_1a_2\ldots a_i}(\phi)|0\rangle, \tag{2.126}$$

where $1 \leq i \leq N(N-1)/2$.

One can proceed to compute $\mathcal{H}_G$ for any G in a similar manner. For brevity, we shall leave this to the interested reader.

2.4.1 The Nonperturbative Partition Function

We shall now furnish a general prescription that will allow us to compute, nonperturbatively, the partition function of our model on any three-manifold with target space X.

Suppose we have manifolds M_1 and M_2 whose boundaries are the same compact Riemann surface Σ but with opposite orientations, such that after gluing them along Σ, we get a new manifold M. Then, from the axioms of quantum field theory, the partition function on M with target space X would be given by

$$Z_X(M) = \langle M_2 | M_1 \rangle. \tag{2.127}$$

Here, $|M_1\rangle \in \mathcal{H}_1$ is a state due to the path integral over M_1 that is associated with Σ, and $|M_2\rangle \in \mathcal{H}_2$ is a state due to the path integral over M_2 that is also associated with Σ, where the Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ are canonically dual to each other.

We could also twist the boundary of M_1 by an element U of the mapping class group of Σ prior to gluing, whence the partition function on the resulting three-manifold M^U would be given by

$$Z_X(M^U) = \langle M_2 | \hat{U} | M_1 \rangle, \tag{2.128}$$

where $\hat{U}$ is an operator acting in $\mathcal{H}_1$ that represents U. In this manner, one can, with appropriate choices of Σ, M_1 and M_2, construct any three-manifold M^U, and upon determining how $\hat{U}$ acts on $|M_1\rangle$ to produce another state in $\mathcal{H}_1$, the corresponding partition function on M^U can be determined via a tractable calculation on M. Therefore, let us determine the action of $\hat{U}$ on $|M_1\rangle$.

For concreteness, let us consider $\Sigma = \mathbf{T}^2$ whence U is an element of $SL(2,\mathbb{Z})$, and M_1 is a solid torus. Then, we can conveniently choose on Σ, basic one-forms $\xi^{1,2}$ and basic one-cycles $C_{1,2}$, whereby

$$\int_{C_b} \xi^a = \delta_{ab}, \tag{2.129}$$

so that the matrix

$$U = \begin{pmatrix} p & q \\ r & s \end{pmatrix} \in SL(2,\mathbb{Z}), \quad ps - qr = 1, \tag{2.130}$$

transforms the pair of cycles $(C_1,\ C_2)$ as

$$U: \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} \to \begin{pmatrix} p & q \\ r & s \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}. \tag{2.131}$$

Now, from (2.107), we have

$$|M_1\rangle = |\psi_0\rangle \otimes \tilde{\Phi}_1 |0\rangle, \tag{2.132}$$

where the subscripts '0' and '1' accompanying $|\psi\rangle$ and $\tilde{\Phi}$ are just convenient labels to associate them to $|M_1\rangle$.

Let us first determine how $\hat{U}$ acts on $|\psi_0\rangle$. From the explanation of CS theory in [3], since $|\psi_0\rangle$ is associated with a path integral on M_1 with no operator insertions (see (2.108)), we can regard it as a vector v_0 in the Verlinde basis of $\mathcal{H}_{\text{CS}}(A_0^\vartheta, \mathbf{T}^2)$, the space of integrable representations of the affine algebra associated with G at level k_{cs}, where the subscript '0' in v_0 means that it is associated with the trivial representation of G [3, section 4.3]. As such, according to *loc. cit.*, we have

$$\hat{U}|\psi_0\rangle = K_0{}^j |\psi_j\rangle, \tag{2.133}$$

where $|\psi_j\rangle$ corresponds to the vector v_j in $\mathcal{H}_{\text{CS}}(A_0^\vartheta, \mathbf{T}^2)$ that is associated with the R_j representation of G; the R_j's are in one-to-one correspondence with the highest weights of G; K is the Verlinde matrix [49]; and the underlying wave function is

$$\psi_j(A_0^\vartheta) = \int_{A|_{\mathbf{T}^2} = A_0^\vartheta} DA\ W_j(C)\, e^{-S_{cs}}, \tag{2.134}$$

where

$$W_j(C) = \text{Tr}_{R_j} P \exp\left(\oint_C A^a \mu_{+a} \right) \tag{2.135}$$

is the Q-closed (and therefore gauge-invariant) trace of the holonomy of the one-form $A^a \mu_{+a}$ along the longitudinal cycle C in M_1 taken in the representation R_j.[7]

[7] Note that the arguments in [3, section 4.3] involve the Wilson loop operator $\mathscr{W}_j(C) = \text{Tr}_{R_j} P \exp\left(\oint_C A^a T_a\right)$ and not $W_j(C) = \text{Tr}_{R_j} P \exp\left(\oint_C A^a \mu_{+a}\right)$, where the T_a's are generators of the Lie algebra $\mathfrak{g}$ of G. Nevertheless, recall that $d\mu_{+a} = -i_{V_a}(\Omega)$, where the vector fields V_a associated with the G-action on X are generators of $\mathfrak{g}$; in other words, like the T_a's, the μ_{+a}'s can be labeled by representations of G. Also, under a gauge transformation with parameter Λ, we have $\delta_\Lambda(\mu_{+a}) = -f_{ac}^d \Lambda_d \mu_+^c$ and $\delta_\Lambda T_a = f_{ac}^d \Lambda_d T^c$. Last but not least, we have the Poisson

Next, let us determine the action of $\hat{U}$ on $\tilde{\Phi}_1$, where $\tilde{\Phi}_1$ takes the generic form in (2.117). As $\eta^{\bar{I}}$ is a (geometrically-trivial) scalar on Σ, we only need to consider the action on χ^I_α.

Recall that we can write

$$\chi^I_\mu = \chi^I_\alpha \omega^\alpha_\mu(A^\vartheta_0), \tag{2.136}$$

where the ω^α's are covariant harmonic one-forms on Σ. This is similar to the case in RW theory, except that here, the ω^α's also depend on a certain flat connection background A^ϑ_0. Since we are free to choose A^ϑ_0, let us choose a background whereby there are two covariant harmonic forms on Σ, i.e., there are two solutions to (2.93). Then, the fermionic zero modes can be expressed as

$$\chi^I_\beta = \int_{C'_\beta} \chi^I_\alpha \omega^\alpha, \quad \alpha, \beta = 1, 2, \tag{2.137}$$

since

$$\int_{C'_\beta} \omega^\alpha = \delta^\alpha_\beta, \tag{2.138}$$

where C'_β are a pair of covariant basic one-cycles in $\mathbf{T}^2$. Assuming that our background is also such that $C'_{1,2}$ is not deformed away from $C_{1,2}$, we also have

$$U : \begin{pmatrix} C'_1 \\ C'_2 \end{pmatrix} \to \begin{pmatrix} p & q \\ r & s \end{pmatrix} \begin{pmatrix} C'_1 \\ C'_2 \end{pmatrix}. \tag{2.139}$$

Then, according to (2.138) and (2.137),

$$\hat{U} : \begin{pmatrix} \chi^I_1 \\ \chi^I_2 \end{pmatrix} \to \begin{pmatrix} p & q \\ r & s \end{pmatrix} \begin{pmatrix} \chi^I_1 \\ \chi^I_2 \end{pmatrix}, \tag{2.140}$$

where we shall regard χ^I_2 to be the annihilation operator. Therefore,

$$\hat{U} : \tilde{\Phi}_1(\phi, \eta, \chi^I_1) \to \tilde{\Phi}_1(\phi, \eta, p\chi^I_1 + q\chi^I_2). \tag{2.141}$$

We would like to emphasize that the modular transformation will not modify the

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bracket relation (2.2). Altogether, this means that we can, for all our purposes, regard μ_{+a} as the matrix T_a whence we can also regard W_j as a Wilson loop operator $\mathscr{W}_j$.

intrinsic definition of the zero modes χ, ϕ and η (which depends on the respective differential operators covariant with respect to A_0^ϑ).[8] Hence, the (generic) intrinsic definition (2.117) of $\tilde{\Phi}_1$ will not be modified either. Thus, there is no ambiguity in the map (2.141).

At any rate, the property that the vacuum $|0\rangle$ would be annihilated by χ_2^I must also hold after a transformation by $\hat{U}$; in other words, if

$$\chi_2^I|0\rangle = 0, \tag{2.142}$$

then

$$\hat{U} : |0\rangle \rightarrow |0\rangle', \tag{2.143}$$

where

$$(r\chi_1^I + s\chi_2^I)|0\rangle' = 0. \tag{2.144}$$

Since (2.94) means that we can represent χ_1^I by multiplication and χ_2^I by $\Omega^{IJ}\partial/\partial\chi_1^J$, we can also write (2.144) as

$$\left(s\Omega^{IJ}\frac{\partial}{\partial\chi_1^J} + r\chi_1^I\right)|0\rangle' = 0, \tag{2.145}$$

which implies that

$$|0\rangle' = \exp\left(-\frac{r}{s}\Omega_{IJ}\chi_1^I\chi_1^J\right)|0\rangle. \tag{2.146}$$

In all, this means that $\hat{U}$ acts on $|M_1\rangle$ as

$$\hat{U} : |M_1\rangle \rightarrow |M_1\rangle', \tag{2.147}$$

[8]This claim can be justified as follows. First, note that A_0^ϑ, being flat, is a covariant harmonic one-form on Σ, just like χ^I; hence, its components A_{01}^ϑ and A_{02}^ϑ will transform as in (2.140). Nevertheless, the covariant equation $\Omega_{IJ}\epsilon^{\tau\mu\nu}\partial_\mu\chi_\nu^J + \Omega_{IK}\epsilon^{\tau\mu\nu}\partial_J V_a^K A_{0\mu}^a\chi_\nu^J = 0$; $\mu,\nu = 1,2$, which defines χ, is a scalar equation on Σ (since the μ and ν indices are fully contracted) – it is thus insensitive to the transformation of Σ by U whence the definition of χ would be unmodified. As for ϕ and η, they are defined by the following one-form equations on Σ (since there is a free μ index): $\partial_\mu\phi + A_\mu^a V_a = 0$ and $g_{I\bar{J}}\partial^\mu\eta^{\bar{J}} + g_{I\bar{K}}\partial_{\bar{J}}V_a^{\bar{K}}A_0^{a\mu}\eta^{\bar{J}} = 0$. If we rewrite these equations as $D_\mu\phi = 0$ and $\mathcal{D}^\mu\eta = 0$, $\mu = 1,2$, then the action of U on Σ would map the first equation from $D_1\phi = 0 \rightarrow pD_1\phi + qD_2\phi = 0$ and $D_2\phi = 0 \rightarrow rD_1\phi + sD_2\phi = 0$. But $D_1\phi$ and $D_2\phi$ are independent quantities whence $pD_1\phi + qD_2\phi = 0$ and $rD_1\phi + sD_2\phi = 0$ imply that $D_\mu\phi = 0$, $\mu = 1,2$, which is the same as the original equation. The same argument applies for the second equation involving η. Hence, the definition of ϕ and η would also be unmodified.

where

$$|M_1\rangle' = K_0{}^j|\psi_j\rangle \otimes \tilde{\Phi}_1\left(\phi, \eta, p\chi_1^I + q\Omega^{IJ}\frac{\partial}{\partial\chi_1^J}\right)\cdot \exp\left(-\frac{r}{s}\Omega_{IJ}\chi_1^I\chi_1^J\right)|0\rangle. \quad (2.148)$$

We are now ready to compute the partition function $Z_X(M^U)$. If

$$\langle M_2| = \langle 0|\tilde{\Phi}_2^\dagger(\phi, \eta, \chi_1^I) \otimes \langle\gamma|, \quad (2.149)$$

then $Z_X(M^U) = \langle M_2|\hat{U}|M_1\rangle = \langle M_2|M_1\rangle'$ can also be expressed as

$$Z_X(M^U) = K_0{}^j\langle\gamma|\psi_j\rangle\cdot\langle 0|\tilde{\Phi}_2^\dagger(\phi,\eta,\chi_1^I)\,\tilde{\Phi}_1(\phi,\eta,p\chi_1^I{+}q\Omega^{IJ}\partial/\partial\chi_1^J)\exp(-r\Omega_{IJ}\chi_1^I\chi_1^J/s)|0\rangle. \quad (2.150)$$

Notice that $\langle\gamma|\psi_j\rangle$ is just the CS path integral on M with an insertion of the Wilson loop operator $W_j(C)$ along the trivial knot C, i.e., it is the topologically-invariant expectation value $\langle W_j(C)\rangle_{\mathrm{CS}}$ of $W_j(C)$ in CS theory on M (see also footnote 7).

Notice also that $\langle 0|\tilde{\Phi}_2^\dagger(\phi,\eta,\chi_1^I)\,\tilde{\Phi}_1(\phi,\eta,p\chi_1^I{+}q\Omega^{IJ}\partial/\partial\chi_1^J)\exp(-r\Omega_{IJ}\chi_1^I\chi_1^J/s)|0\rangle$ can be expressed as the scalar product $\langle\Phi^{(2)}|\Phi^{(1)}\rangle$ of the non-dynamically gauged RW theory on M with target X, where $|\Phi^{(2)}\rangle = \tilde{\Phi}_2(\phi,\eta,\chi_1^I)|0\rangle$, and $|\Phi^{(1)}\rangle = \tilde{\Phi}_1(\phi,\eta,p\chi_1^I{+}q\Omega^{IJ}\partial/\partial\chi_1^J)\cdot\exp(-r\Omega_{IJ}\chi_1^I\chi_1^J/s)|0\rangle = \tilde{\Phi}_1'(\phi,\eta,\chi_1^I)|0\rangle$. Since $\tilde{\Phi}_1'(\phi,\eta,\chi_1^I)$ and $\tilde{\Phi}_2(\phi,\eta,\chi_1^I)$ correspond to classes in $H_G(\mathcal{M}^\vartheta)$, the scalar product $\langle\Phi^{(2)}|\Phi^{(1)}\rangle = \langle 0|\tilde{\Phi}_2^\dagger\tilde{\Phi}_1'|0\rangle$ can be computed via G-equivariant Poincáre duality as an intersection number $(\tilde{\Phi}_2, \tilde{\Phi}_1')_{\mathcal{M}^\vartheta}$ of G-equivariant cycles in $\mathcal{M}^\vartheta$ that are dual to $\tilde{\Phi}_2$ and $\tilde{\Phi}_1'$, respectively.

Therefore, we can actually write

$$\boxed{Z_X(M^U) = K_0{}^j\langle W_j(C)\rangle_{\mathrm{CS}(M)}\cdot(\tilde{\Phi}_2, \tilde{\Phi}_1')_{\mathcal{M}^\vartheta(M,X)}} \quad (2.151)$$

As claimed, the partition function on M^U can be calculated in terms of well-defined quantities on M. In fact, it can be expressed as a product of a CS and an equivariant RW topological invariant of M!

2.5 New Knot Invariants From Supersymmetric Wilson Loops

As mentioned in subsection 4.2, a Q-invariant (and therefore gauge-invariant) Wilson loop operator along a knot $\mathcal{K} \subset M$ can be constructed as[9]

$$W_R(\mathcal{K}) = \mathrm{Tr}_R P \exp\left(\oint_{\mathcal{K}} A^a \mu_{+a}\right), \tag{2.152}$$

where M is an arbitrary Riemannian three-manifold, and R denotes the representation R of the Lie group G which acts on the hyperkähler target space X. The trace Tr is taken over G whose Lie algebra $\mathfrak{g}$ is generated by the μ_{+a}'s in the representation R (see footnote 7).

The Canonical Formalism

In the canonical formalism of section 4, where we restrict ourselves to the zero modes of the fields in the region $\Sigma \times I \subset M$ of interest, we have, in the absence of the Wilson loop operator $W_R(\mathcal{K})$, the "Gauss Law" constraint $\delta L/\delta A_\tau = 0$:

$$F^a_{\mu\nu} = 0, \tag{2.153}$$

where τ and $\{\mu, \nu\}$ are the coordinates on I and Σ, respectively.

If we include in our theory, multiple copies of the Wilson loop operator $W_{R_1}(\mathcal{K}_1)W_{R_2}(\mathcal{K}_2)\cdots$ in the representations $R_1, R_2, \ldots$ of G, the "Gauss Law" constraint becomes

$$F^a_{\mu\nu} = \epsilon_{\mu\nu} \sum_s \delta^2(x - P_s)\mu^a_{+(s)}. \tag{2.154}$$

Here, the P_s's are the points on Σ that the knots $\mathcal{K}_1, \mathcal{K}_2, \ldots$ intersect, and they are labeled by the representations R_s via the $\mu^a_{+(s)}$'s (see footnote 7).

The physical Hilbert space $\mathcal{H}_{\Sigma,P_s,R_s}$ of our theory can then be obtained by quantizing the underlying symplectic phase space $\mathscr{M}$ determined by the $A_\mu, \eta, \chi_\tau, \chi_\mu$ fields, their respective momentum conjugate A_ν, χ_τ, η, χ_ν (computed in (2.92)), the constraint (2.154), and the conditions $D\phi = 0$, $D\chi = 0$ and $D\eta = 0$. Specifically, according to the theory of geometric quantization [50], $\mathcal{H}_{\Sigma,P_s,R_s}$ would correspond to the space $H^0(\mathscr{L}, \mathscr{M})$ of holomorphic sections of

[9]Note that we have used the relation $\mu^a_+\mu_{+a} = 0$ to construct the following expression.

a certain line bundle $\mathscr{L}$, where the curvature of $\mathscr{L}$ is given by $\sqrt{-1}$ times the symplectic two-form of $\mathscr{M}$.

The Path Integral Formalism and New Knot Invariants

Now that we have furnished, through the canonical formalism perspective, a formal description of the physical Hilbert space of the theory in the presence of multiple Wilson loop operators $W_{R_j}(\mathcal{K}_j)$, let us compute explicitly the expectation value of such Wilson loop operators which will provide us with *new* knot invariants of three-manifolds.

To compute the expectation value $\langle W_R(\mathcal{K})\rangle$ via the path integral, we will need to replace $W_R(\mathcal{K})$ in (2.152) with its gauge-fixed version. To this end, recall that after gauge-fixing, Q would be replaced by $\hat{Q} = Q + Q_{\mathrm{FP}}$. Of course, $W_R(\mathcal{K})$ in (2.152) is no longer invariant under the field transformations generated by $\hat{Q}$. Nevertheless, a $\hat{Q}$-invariant gauge-fixed replacement can be constructed as

$$\boxed{\tilde{W}_R(\mathcal{K}) = \mathrm{Tr}_R P \exp \oint_{\mathcal{K}} \mathbb{A} := \mathrm{Tr}_R P \exp \left(\oint_{\mathcal{K}} (A_a \mu^a_+ + \chi^J \partial_J \mu^a_+ c_a - \frac{1}{2} f_{abd} A^a c^b c^d) \right)} \tag{2.155}$$

One can show that

$$\delta_{\hat{Q}} \oint \left(A_a \mu^a_+ + \chi^J \partial_J \mu^a_+ c_a - \frac{1}{2} f_{abd} A^a c^b c^d \right) = \oint d \left(c_a \mu^a_+ - \frac{1}{6} f_{abd} c^a c^b c^d \right) = 0, \tag{2.156}$$

so

$$\delta_{\hat{Q}} \tilde{W}_R(\mathcal{K}) = 0, \tag{2.157}$$

as claimed.

To compute perturbatively the following expectation value of multiple Wilson loops

$$\langle \prod_j \tilde{W}_{R_j}(\mathcal{K}_j) \rangle = \int D\phi DA D\eta D\chi Dc D\bar{c}\; e^{-S} \prod_j \tilde{W}_{R_j}(\mathcal{K}_j), \tag{2.158}$$

(where the Lagrange multiplier field B has already been integrated out to give the gauge-fixing condition $\partial^\mu A^a_\mu = 0$), we first expand each $\tilde{W}_j$ around the flat connection A_0 and the covariantly constant map ϕ_0 as

$$\tilde{W}_{R_j}(\mathcal{K}_j) = \mathrm{Tr}_{R_j} P \exp \int_{\mathcal{K}_j} \mathbb{A} = \mathrm{Tr}_{R_j} P \exp \int_{\mathcal{K}_i} \Big\{ A_{0a}\mu_+^a(\phi_0) + \tilde{A}_a \mu_+^a(\phi_0) + \tilde{A}_a \partial_J \mu_+^a(\phi_0)\varphi^J + \cdots$$
$$+ \chi^J \partial_J \mu_+^a(\phi_0) c_a + \chi^J \partial_K \partial_J \mu_+^a(\phi_0)\varphi^K c_a + \cdots$$
$$-\frac{1}{2}(f_{abd}A_0^a c^b c^d + f_{abd}\tilde{A}^a c^b c^d)\Big\}, \tag{2.159}$$

where $\tilde{A}$ and φ are fluctuations around A_0 and ϕ_0, and '$\cdots$' denotes all other expansion terms around ϕ_0. Notice that the above path-ordered exponential can also be expressed as

$$\begin{aligned}\mathrm{Tr}_{R_j} P \exp \int_{\mathcal{K}_j} \mathbb{A} =& \mathrm{Tr}_{R_j} \left(\exp \int_{\mathcal{K}_j} A_0 \mu_+(\phi_0) \right) \\ & \times [1 \\ & + \mathrm{Tr}_{R_j} \int_{\mathcal{K}_j} \left(\tilde{A}\mu_+(\phi_0) + \tilde{A}\partial_J \mu_+(\phi_0)\varphi^J + \chi^J \partial_J \mu_+(\phi_0) c - \frac{1}{2}(A_0 cc + \tilde{A}cc) + \cdots \right) \\ & + \mathrm{Tr}_{R_j} \int_{\mathcal{K}_j \times \mathcal{K}_j} \left(\tilde{A}\mu_+(\phi_0) + \tilde{A}\partial_J \mu_+(\phi_0)\varphi^J + \chi^J \partial_J \mu_+(\phi_0) c - \frac{1}{2}(A_0 cc + \tilde{A}cc) + \cdots \right)^2 \\ & + \cdots],\end{aligned} \tag{2.160}$$

where we have and shall henceforth omit the Lie algebra index for notational simplicity. Note that because of (2.3), any term in the correlation function which contains $\mathrm{Tr}_{R_j}\left(\mu_+^2(\phi_0)\right)$ is automatically zero.[10]

After performing the expansion, we can evaluate the correlation function of $\prod_j \tilde{W}_{k_j}(\mathcal{K}_j)$ by the same method used to evaluate the partition function in section 3. Because of (2.160), we can, like in (2.76), express the correlation function as

$$\langle \prod_j \tilde{W}_{K_j}(\mathcal{K}_j) \rangle = \sum_{A_0^\vartheta} \left(e^{-\int_M k_{cs} L_{cs}(A_0^\vartheta)} \cdot Z_0(A_0^\vartheta) \cdot \prod_j \mathrm{Tr}_{R_j} e^{\int_{\mathcal{K}_j} A_0^\vartheta \mu_+(\phi_0)} \right) \mathfrak{W}(M, X, G; A_0^\vartheta; k_{cs}), \tag{2.162}$$

[10] For example, according to (2.3), the correlation function

$$\langle \mathrm{Tr}_{R_j}(\tilde{A}\mu_+(\phi_0)\tilde{A}\mu_+(\phi_0)) \rangle \sim \langle \mathrm{Tr}_{R_j}(\mu_+^2(\phi_0)) \rangle = 0 \tag{2.161}$$

for the classical configuration ϕ_0.

where the first factor in parenthesis is manifestly topologically-invariant – $e^{-\int_M k_{cs} L_{cs}(A_0^\vartheta)}$ is the topological factor coming from the Chern-Simons part of the total Lagrangian evaluated at a flat connection A_0^ϑ, $Z_0(A_0^\vartheta)$ is the topological one-loop contribution given in (2.46), and $\prod_j \mathrm{Tr}_{R_j} e^{\int_{\mathcal{K}_j} A_0^\vartheta \mu_+(\phi_0)}$ is the product of noninteracting topological Wilson loops – and

$$\mathfrak{W}(M, X, G; A_0^\vartheta; k_{cs}) = \sum_\Gamma \mathfrak{W}_\Gamma(M, X, G; A_0^\vartheta; k_{cs}^m). \tag{2.163}$$

Here, $\sum_\Gamma$ is a sum over all possible Feynman diagrams with two or more loops that (i) have the right number of fermionic zero modes to absorb those that appear in the path integral measure, and (ii) are free of the coupling constant k. The label k_{cs}^m (where m may vanish) means that Γ carries with it a factor of k_{cs}^m.

Note that we have two types of Feynman diagrams here. The first type is where the vertices of $\tilde{W}_{k_j}(\mathcal{K}_j)$ do not contract with the vertices of the Lagrangian L; let us denote this type of diagrams as Γ^*. The second type is where the vertices of $\tilde{W}_{k_j}(\mathcal{K}_j)$ contract with the vertices of L; let us denote this type of diagrams as $\Gamma^\diamond$. In other words, we can write the total expectation value as

$$\langle \prod_j \tilde{W}_{K_j}(\mathcal{K}_j) \rangle = \langle \prod_j \tilde{W}_{R_j}(\mathcal{K}_j) \rangle_{\Gamma^*} + \langle \prod_j \tilde{W}_{R_j}(\mathcal{K}_j) \rangle_{\Gamma^\diamond}. \tag{2.164}$$

Because the total expectation value $\langle \prod_j \tilde{W}_{K_j}(\mathcal{K}_j) \rangle$ is topologically-invariant at the outset, we have

$$\frac{\delta \langle \prod_j \tilde{W}_{K_j}(\mathcal{K}_j) \rangle}{\delta h_{\mu\nu}} = \frac{\delta \langle \prod_j \tilde{W}_{K_j}(\mathcal{K}_j) \rangle_{\Gamma^*}}{\delta h_{\mu\nu}} + \frac{\delta \langle \prod_j \tilde{W}_{K_j}(\mathcal{K}_j) \rangle_{\Gamma^\diamond}}{\delta h_{\mu\nu}} = 0, \tag{2.165}$$

where $h_{\mu\nu}$ is the metric of M.

Similar to CS and RW theory, because the propagators are not topologically-invariant, each diagram in $\langle \prod_j \tilde{W}_{K_j}(\mathcal{K}_j) \rangle$ is not topologically-invariant by itself. However, the total expectation value is still topological because the variations (under a change in $h_{\mu\nu}$) of the diagrams cancel themselves out exactly.

In our case, notice that for the diagrams Γ^*, we only have the propagator factors

$$\int_{M \times M} d^3 x_l d^3 y_l \ \Delta(x_l, y_l) \tag{2.166}$$

and

$$\int_{\mathcal{K}_i \times \mathcal{K}_j} dx_l dy_l \; \Delta(x_l, y_l), \tag{2.167}$$

while for the diagrams $\Gamma^\diamond$, we *also* have the propagator factor

$$\int_M d^3 x_l \int_{\mathcal{K}_j} dy_l \; \Delta(x_l, y_l) \tag{2.168}$$

coming from the contractions between the vertices of $\tilde{W}_{k_j}(\mathcal{K}_j)$ and that of the Lagrangian L. This means that the variations of the Γ^* diagrams cannot cancel out the variations of the $\Gamma^\diamond$ diagrams. In turn, this and (2.165) imply that

$$\frac{\delta \langle \prod_j \tilde{W}_{K_j}(\mathcal{K}_j) \rangle_{\Gamma^*}}{\delta h_{\mu\nu}} = 0 \quad \text{and} \quad \frac{\delta \langle \prod_j \tilde{W}_{K_j}(\mathcal{K}_j) \rangle_{\Gamma^\diamond}}{\delta h_{\mu\nu}} = 0 \tag{2.169}$$

simultaneously. In other words, both $\langle \prod_j \tilde{W}_{R_j}(\mathcal{K}_j) \rangle_{\Gamma^*}$ and $\langle \prod_j \tilde{W}_{R_j}(\mathcal{K}_j) \rangle_{\Gamma^\diamond}$ are *independently* topologically-invariant.

For brevity, let us henceforth focus our discussion on $\langle \prod_j \tilde{W}_{K_j}(\mathcal{K}_j) \rangle_{\Gamma^*}$. For the diagrams Γ^*, we can write

$$\mathfrak{W}_{\Gamma^*}(M, X, G; A_0^\vartheta; k_{cs}^m) = \int_{\mathcal{M}^\vartheta} \sqrt{g}\, d^{2n}\phi_0^I d^{2n}\phi_0^{\bar{I}} \; W_{\Gamma^*}(X, G; \phi_0, A_0^\vartheta) f_{\Gamma^*}(X, G; \phi_0, A_0^\vartheta) I'_{\Gamma^*}(M; A_0^\vartheta; k_{cs}^m) \prod_j \Gamma^*_{\tilde{W}_j}, \tag{2.170}$$

where the functions W_{Γ^*}, f_{Γ^*} and I'_{Γ^*} are similar to those in (2.89)–(2.90) as they result solely from contractions among the vertices coming from L, and $\Gamma^*_{\tilde{W}_j}$ denotes the contribution of $\tilde{W}_{R_j}(\mathcal{K}_j)$ to each $\mathfrak{W}_{\Gamma^*}$.

According to the discussion leading up to (2.88), we can also write[11]

$$\Gamma^*_{W_j} = W_{\Gamma^*,W_j}(X,G;\phi_0,A_0^\vartheta)\, f_{\Gamma^*,W_j}(X,G;\phi_0,A_0^\vartheta)\, I'_{\Gamma^*,W_j}(M;A_0^\vartheta). \qquad (2.173)$$

Thus, we have

$$\mathfrak{W}_{\Gamma^*}(M,X,G;A_0^\vartheta;k_{cs}^m) = \mathscr{W}_{\Gamma^*}(X,G;A_0^\vartheta)\, \mathscr{I}'_{\Gamma^*}(M;A_0^\vartheta;k_{cs}^m), \qquad (2.174)$$

where

$$\boxed{\mathscr{W}_{\Gamma}(X,G;A_0^\vartheta) = \int_{\mathcal{M}^\vartheta} \sqrt{g}\, d^{2n}\phi_0^I\, d^{2n}\phi_0^{\bar{I}}\ (W_{\Gamma^*} f_{\Gamma^*} \prod_j W_{\Gamma^*,W_j} f_{\Gamma^*,W_j})(X,G;\phi_0,A_0^\vartheta)} \qquad (2.175)$$

can be regarded as a weight factor which combines the structural information of the hyperkähler manifold X and the Lie algebra $\mathfrak{g}$ of the gauge group G, and

$$\boxed{\mathscr{I}'_{\Gamma^*}(M;A_0^\vartheta;k_{cs}^m) = (I'_{\Gamma^*} \prod_j I'_{\Gamma^*,W_j})(M;A_0^\vartheta;k_{cs}^m)} \qquad (2.176)$$

Therefore,

$$\langle \prod_j \tilde{W}_{R_j}(\mathcal{K}_j)\rangle_{\Gamma^*} = \sum_{A_0^\vartheta}\sum_{\Gamma^*}\left(e^{-\int_M k_{cs}L_{cs}(A_0^\vartheta)} \cdot Z_0 \cdot \prod_j \mathrm{Tr}_{R_j} e^{\int_{\mathcal{K}_j} A_0^\vartheta \mu_+(\phi_0)}\right) \mathscr{W}_{\Gamma^*}(X,G;A_0^\vartheta)\, \mathscr{I}'_{\Gamma^*}(M;A_0^\vartheta;k_{cs}^m), \qquad (2.177)$$

[11]For example, consider the contribution $\Gamma^*_{\tilde{W}_j,A\varphi}$ from the term

$$\mathrm{Tr}_{R_j}\int_{\mathcal{K}_j\times\mathcal{K}_j} \tilde{A}\partial_I\mu_+\varphi^I \tilde{A}\partial_J\mu_+\varphi^J \qquad (2.171)$$

in (2.160) (assuming that the fermionic zero modes in the measure have been absorbed exactly by the vertices from L which accompany this term). Performing the contraction, we get

$$\begin{aligned}
\Gamma^*_{\tilde{W}_j,A\varphi} &= \mathrm{Tr}_{R_j}\int_{\mathcal{K}_j\times\mathcal{K}_j} dx^\mu dx^\nu\ \partial_I\mu_+\partial_J\mu_+\tilde{A}_\mu\tilde{A}_\nu\varphi^I\varphi^J \\
&= \mathrm{Tr}_{R_j}\int_{\mathcal{K}_j\times\mathcal{K}_j} dx^\mu dx^\nu\ \partial_I\mu_+\partial_J\mu_+\Delta_\mu^{(A\varphi)I}\Delta_\nu^{(A\varphi)J} \\
&= \mathrm{Tr}_{R_j}\left(\partial_I\mu_+\partial_J\mu_+\ f^I(X,G)f^J(X,G)\ \int_{\mathcal{K}_j\times\mathcal{K}_j} dx^\mu dx^\nu \Delta_\mu^{'(A\varphi)}\Delta_\nu^{'(A\varphi)}\right) \\
&= W_{\Gamma^*_{W_j,A\varphi}}(X,G;\phi_0,A_0^\vartheta)\, f_{\Gamma^*_{W_j,A\varphi}}(X,G;\phi_0,A_0^\vartheta)\, I'_{\Gamma^*_{W_j,A\varphi}}(M;A_0^\vartheta),
\end{aligned} \qquad (2.172)$$

where μ,ν run over all directions in M, and the second-last equality is due to (2.65).

where

$$\boxed{\langle\prod_j \tilde{W}_{R_j}(\mathcal{K}_j)\rangle_{\Gamma^*,A_0^\vartheta} = \sum_{\Gamma^*} \mathscr{W}_{\Gamma^*}(X,G;A_0^\vartheta)\,\mathscr{I}'_{\Gamma^*}(M;A_0^\vartheta;k_{cs}^m)} \tag{2.178}$$

is a knot invariant of three-manifolds which depends on *both* G and X, that also defines a knot weight system whose weights $\mathscr{W}_{\Gamma^*}(X,G;A_0^\vartheta)$ are characterized by both Lie algebra structure *and* hyperkähler geometry.

Chapter 3

$\mathcal{N} = 2$ Supersymmetric Gauge Theories and Quantum Integrable Systems

3.1 Introduction

We have seen in the previous chapter that both the perturbative and nonperturbative computations of a three-dimensional supersymmetric gauge theory – a topological Chern-Simons sigma model – resulted in interesting and significant applications in studying three-manifold invariants. In this chapter, we shift gears to study a different type of theory: four-dimensional (partially) topologically twisted $\mathcal{N} = 2$ supersymmetric gauge theories. It turns out that our study brings about an intriguing correspondence between four-dimensional $\mathcal{N} = 2$ supersymmetric gauge theories in the low energy limit and quantum integrable systems. Before revealing our story, let us review the background that our work has its roots in.

3.1.1 Seiberg-Witten Theory

Here we would like to review Seiberg-Witten theory, the effective theory of four-dimensional $\mathcal{N}= 2$ super-Yang-Mills theory in the low energy limit. We shall see that Seiberg-Witten theory can be described by a complex integrable system.

But before proceeding, here one may naturally ask one question: what is the profit of dealing with only the low energy limit? The answer is as follows. Despite the constraints imposed by the supersymmetries, the $\mathcal{N} = 2$ supersymmetric gauge theories are unlikely to be exactly solvable, in the sense of computing all its correlation functions exactly. In our favour, many physically interesting questions have to do with the non-perturbative dynamics of the theories, and can be answered by understanding the properties of the vacuum and the lowest energy excitations above the vacuum. Thus, understanding the theories in the low energy limit is important and interesting on its own.

To this end, one would like to construct the effective action for the low energy theory. There are two types of effective actions. One is the standard generating functional $\Gamma(\phi)$ of one-particle irreducible Feynman diagrams. It is obtained from the renormalised connected generating functional $W(J)$ by a Legendre transformation:

$$Z(J) = \int D\phi e^{-S(\phi)+J\cdot\phi} = e^{-W(J)} \equiv e^{-\Gamma(\phi)+J\cdot\phi}, \tag{3.1}$$

where $\Gamma(\phi) \equiv \Gamma(\phi, \mu)$ also depends on the energy scale μ used to define the β-function (i.e. μ is a UV-cutoff), i.e. the renormalization group equation:

$$\beta(g) = \frac{\partial g}{\partial(\log\mu)}. \tag{3.2}$$

In contrast, in defining the other type of effective action, we expand the fields by two parts

$$\phi = \phi_0 + \tilde{\phi}, \tag{3.3}$$

where ϕ_0 stand for the modes whose energy eigenvalues are below μ, while $\tilde{\phi}$ denote the modes whose energy eigenvalues are above μ. Given this, the partition function can be written as

$$Z = \int D\phi e^{-S(\phi)} = \int D\phi_0 (\int D\tilde{\phi} e^{-S(\phi_0+\tilde{\phi})}) = \int D\phi_0 e^{-S_{\text{eff}}(\phi_0)}. \tag{3.4}$$

Then it is clear that one can define the low energy effective action $S_{\text{eff}}(\phi_0) \equiv S_{\text{eff}}(\phi_0, \mu)$, with all the high energy modes above μ integrated out in the path integral. We call this type of effective action the Wilsonian effective action.

What Seiberg and Witten achieved in [14] is that they determined the Wilsonian effective action for the $\mathcal{N} = 2$ super-Yang-Mills theory on $\mathbb{R}^4$ with gauge group $SU(2)$ (later we will generalize $SU(2)$ to a general gauge group $\mathcal{G}$ of rank r). Such a Wilsonian effective theory is a sigma model and its action reads [51]

$$\frac{1}{16}\,\mathrm{Im}\int \mathrm{d}^4x[\int \mathrm{d}^2\theta\mathcal{F}^{''}(\Phi)W^{\alpha}W_{\alpha}+\int \mathrm{d}^2\theta\mathrm{d}^2\bar{\theta}\Phi^{\dagger}\mathcal{F}^{'}(\Phi)], \tag{3.5}$$

where W denotes the superpotential and $\mathcal{N}=2$ prepotential function $\mathcal{F}$ is holomorphic, both of which depend on the superfield Φ. Here the gauge group $SU(2)$ is broken to $U(1)$, while the $\mathcal{N}=2$ supersymmetry remains. For this sigma model, the metric on the moduli space $\mathcal{B}$ takes the following form

$$\mathrm{d}s^2=\mathrm{Im}\,\mathcal{F}^{''}(a)\mathrm{d}a\mathrm{d}\bar{a}=\mathrm{Im}\,\tau(a)\mathrm{d}a\mathrm{d}\bar{a}, \tag{3.6}$$

where $\mathrm{Im}\,\tau(a)$ is the holomorphic effective coupling constant, depending on the moduli space coordinates a and $\bar{a}$ which are the vacuum configurations of the vector multiplet scalar ϕ.

Clearly, if one can determine the moduli space metric (i.e., determine the coupling constant $\tau(a)$) exactly, one can determine the Wilsonian effective action. To this end, one first needs to understand that the prepotential $\mathcal{F}$ is subject to two fundamental physical properties: (i) by the $\mathcal{N}=2$ supersymmetry, $\mathcal{F}(a_i)$ is holomorphic; (ii) $\mathrm{Im}\,\tau(a)\equiv\mathrm{Im}\,\frac{\partial^2\mathcal{F}(a)}{\partial a^2}>0$ by the requirement that the kinetic terms are positive. However, by the property (i), $\mathrm{Im}\,\tau(a)$ is a harmonic function so it cannot have a minimum; and hence (on the compactified complex manifold) the property (ii) cannot be satisfied (i.e. $\mathrm{Im}\,\tau$ can not be bounded from below) anywhere unless $\mathrm{Im}\,\tau(a)$ is a constant (but it is not). Then it follows that the coordinates a, $\bar{a}$ and $\mathcal{F}(a)$ cannot globally satisfy these two properties on the moduli space $\mathcal{B}$. Therefore, the coordinates a, $\bar{a}$ and $\mathcal{F}(a)$ are not appropriate for fully describing the low-energy effective action: a singularity of the metric arises when $\mathrm{Im}\,\tau(a)$ approaches zero.

Since the coordinates a, $\bar{a}$ and $\mathcal{F}(a)$ are only valid in a certain region, in order to globally describe the moduli space, we need to patch the other region of the moduli space $\mathcal{B}$ near a singular point with $\mathrm{Im}\,\tau(a)\to 0$ using a different set of coordinates, which turns out to be provided by the magnetic dual of $\{a,\bar{a}\}$.

The dual coordinates are defined as

$$a_D = \frac{\partial \mathcal{F}}{\partial a},\ \mathrm{d}a_D = \tau \mathrm{d}a. \tag{3.7}$$

Then (3.6) can be written as

$$\mathrm{d}s^2 = \mathrm{Im}(\mathrm{d}a \mathrm{d}\bar{a}_D). \tag{3.8}$$

In this context, if we can determine the monodromy property associated with the singularities on $\mathcal{B}$, that is, determine the following transformation

$$\begin{pmatrix} a \\ a_D \end{pmatrix} \to M_{p_0} \begin{pmatrix} a \\ a_D \end{pmatrix} \tag{3.9}$$

near the singular point p_0 (where M is the transformation matrix), we will be able to describe the moduli space $\mathcal{B}$ globally. So to solve Seiberg-Witten theory, the key point is to solve the monodromies, i.e. to describe the singularities on the moduli space exactly.

For a different region of the moduli space, an appropriate description can be given by a dual effective action. We can define the dual field Φ_D and dual prepotential function $\mathcal{F}(\Phi_D)$ by

$$\Phi = \mathcal{F}^{'}(\Phi),\ \mathcal{F}_D^{'}(\Phi_D) = -\Phi, \tag{3.10}$$

where the duality transformations constitute a Legendre transformation $\mathcal{F}_D(\Phi_D) = \mathcal{F} - \Phi\Phi_D$. By (3.10), we can see

$$\mathcal{F}_D''(\Phi_D) = -\frac{1}{\mathcal{F}''(\Phi)}, \tag{3.11}$$

which shows that the singularity of the metric on the moduli space can be removed by the dual transformation. By such definitions, the action (3.5) can be equivalently written as

$$\frac{1}{16}\,\mathrm{Im}\int \mathrm{d}^4x[\int \mathrm{d}^2\theta \mathcal{F}_D^{''}(\Phi_D) W_D^{\alpha} W_{D\alpha} + \int \mathrm{d}^2\theta \mathrm{d}^2\bar{\theta} \Phi_D^{\dagger} \mathcal{F}_D^{'}(\Phi_D)]. \tag{3.12}$$

More physically, in the $N = 2$ supersymmetric (spontaneously broken) gauge theory context, the transformation (3.9) exchanges the electric and magnetic degrees of freedom, to be more specific, it exchanges the electrically charged states with magnetic monopoles (which are solitons that carry magnetic charge). Let us elaborate on this point as follows. First, as the representations of $\mathcal{N} = 2$ supersymmetry, there are long and short (BPS) multiplets. Here we are interested in the massive states which are in the BPS multiplets. Such states of mass m satisfy the BPS condition $m^2 = |Z|^2$, where Z denotes the central charge of the $\mathcal{N} = 2$ susy algebra. After gauge symmetry breaking, some gauge bosons gain mass through the Higgs mechanism by interacting with the Higgs field, therefore $m = a$, where a are the vacuum configurations of the Higgs boson ϕ. Thus the central charge $Z = a$ by the BPS condition. Generalizing to an arbitrary electrically charged state with integer electric charge n_e, we then have $Z = an_e$. Then, as implied by the electric-magnetic duality, a purely magnetically charged state has $Z = a_D n_m$ with the (integer) magnetic charge n_m. Thus, a dyon, which is a state with both electric and magnetic charges, has $Z = an_e + a_D n_m$.

With this being said, let us go on to formulate Seiberg-Witten theory with a general group $\mathcal{G}$ of rank r in a more mathematical manner (which will help us to see how the complex integrable system arises later). We shall see that the set of a and a_D is indeed suitable for describing the theory, and that the metric $\operatorname{Im}\tau$ we defined is indeed positive definite, giving us a well defined action.

Now we are considering the Coulomb branch of the $\mathcal{N} = 2$ supersymmetric gauge theory with gauge group $\mathcal{G}$ of rank r, which is parametrized by the vacuum expectation values of the gauge-invariant polynomials in the vector multiplet scalar ϕ. On the Coulomb branch, the gauge group is broken to a maximal torus $U(1)^r$, thus ϕ takes value in the abelian subalgebra:

$$\phi = (a_1, a_2, \cdots, a_r). \tag{3.13}$$

We define the complex parameter

$$u_I = \langle \operatorname{Tr} \phi^{(I+1)} \rangle \tag{3.14}$$

to label gauge inequivalent vacua, the manifold of which is just the moduli space $\mathcal{B}$. Hence u_I $(I = 1, \cdots, r)$ are the coordinates on $\mathcal{B}$.

At each point u, $\overrightarrow{a}(u)$ and $\overrightarrow{a_D}(u)$ generate a lattice $\Gamma_u \subset \mathbb{R}^{2r}$ of electric and magentic charges. Then by the Dirac quantization condition, the lattice should be equipped with a nondegenerate skew-symmetric bilinear form

$$\langle\ ,\ \rangle : \Gamma_u \times \Gamma_u \to \mathbb{Z}. \tag{3.15}$$

The charge lattices at the different points of $\mathcal{B}$ form a fibration

$$\Gamma \to \mathcal{B}, \tag{3.16}$$

where the fibration has nontrivial monodromy around the singular loci in $\mathcal{B}$.

Locally on $\mathcal{B}$, one chooses the symplectic basis $\{\alpha_I, \beta^I\} \subset \Gamma$, which satisfies

$$\langle \alpha_I, \alpha_J \rangle = \langle \beta^I, \beta^J \rangle = 0, \langle \alpha_I, \beta^J \rangle = \delta^I_J. \tag{3.17}$$

Such a choice determines a duality frame: a local splitting of Γ into the Lagrangian sublattices Γ_m and Γ_e of magnetic and electric charges.

In this context, for a particle of charge $\gamma \in \Gamma_u$, the central charge $Z_\gamma(u)$ is a holomorphic function on $\mathcal{B}$ satisfying $Z_{\gamma_1+\gamma_2}(u) = Z_{\gamma_1}(u) + Z_{\gamma_2}(u)$, since it is additive. Choosing a symplectic basis, Z can be written as

$$Z = a^I \beta_I + a_{D,I} \alpha^I. \tag{3.18}$$

Then using the property of the symplectic manifold, we can prove that the metric can be positive definite, as follows.

First, we have the nondegeneracy condition $\langle \mathrm{d}Z, \mathrm{d}\bar{Z} \rangle > 0$. Then by (3.17) and (3.18), we get

$$\mathrm{Re}(\mathrm{d}a^I \wedge \mathrm{d}\bar{a}_{D,I}) < 0. \tag{3.19}$$

This, in particular, implies that the matrices $(\partial a^I / \partial u^J)$ and $(\partial a_{D,I} / \partial u^J)$ are invertible for any holomorphic coordinates u^I on $\mathcal{B}$. Thus the a^I give local holomorphic coordinates on $\mathcal{B}$, and so do the $a_{D,I}$. Such coordinates are called special coordinates.

Besides, we have the transversality condition $\langle \mathrm{d}Z, \mathrm{d}Z \rangle = 0$. By (3.17) and (3.18), we obtain

$$\mathrm{d}(a_{D,I}\mathrm{d}a^I) = 0. \tag{3.20}$$

This ensures that locally there is a holomorphic function $\mathcal{F}$ such that $a_{D,I}\mathrm{d}a^I = \mathrm{d}\mathcal{F}$. It is clear that this holomorphic function $\mathcal{F}$ is exactly the aforementioned prepotential, which relates a^I and $a_{D,I}$ by

$$a_{D,I} = \frac{\partial \mathcal{F}}{\partial a^I}. \tag{3.21}$$

The positive (1,1) form $-\mathrm{Re}(\mathrm{d}a^I \wedge \mathrm{d}\bar{a}_{D,I})$ can be interpreted as a Kahler form on $\mathcal{B}$, and the period matrix τ can be defined as

$$\tau_{IJ} = \frac{\partial a_{D,I}}{\partial a^J} = \frac{\partial^2 \mathcal{F}}{\partial a^I \partial a^J}. \tag{3.22}$$

Using this, the nondegeneracy condition (3.19) can be written as

$$(\tau_{IJ} - \bar{\tau}_{IJ})\mathrm{d}a^I \wedge \mathrm{d}\bar{a}^J > 0, \tag{3.23}$$

which leads to the condition

$$\mathrm{Im}\,\tau > 0. \tag{3.24}$$

Therefore, as the metric $\mathrm{Im}\,\tau_{IJ}$ is positive definite, we can define the bosonic part of the effective Lagrangian by

$$\mathcal{L} = \frac{1}{4\pi}\mathrm{Im}\,\tau_{IJ}(\mathrm{d}a^I \wedge \star\mathrm{d}\bar{a}^J + F^I \wedge \star F^J) + \frac{i}{4\pi}\mathrm{Re}\,\tau_{IJ}F^I \wedge F^J, \tag{3.25}$$

where $F^I = \mathrm{d}A^I$ denote the gauge field strengths. Thus, we have successfully constructed the effective theory. As we have seen, in this construction the electric-magnetic duality plays a crucial role. Now, we are ready to show how the complex integral system arises from Seiberg-Witten theory.

3.1.2 Complex Integrable System from Seiberg-Witten Theory

In this subsection, we will show that a complex quantum integrable system naturally arises by using physical properties of Seiberg-Witten theory.

Recall that here the goal is to determine the low-energy effective theory, i.e. determine the prepotential function $\mathcal{F}$ (which is equivalent to determining the metric $\operatorname{Im}\tau$ of the moduli space). Apparently, according to the relation $a_{D,I}\mathrm{d}a^I = \mathrm{d}\mathcal{F}$, this goal can be achieved, at least in principle, if we know $a(u)$ and $a_D(u)$. To determine a and a_D, as we have shown, the key point is to exactly describe the monodromies associated with the singularities on the moduli space. Seiberg and Witten found that the two functions $a(u)$ and $a_D(u)$ with the correct monodromy properties can be described in terms of a family of elliptic curves. As a generalization to Seiberg and Witten's construction, in the following, using a fibration setup, we will be able to describe $a(u)$ and $a_D(u)$ properly. And, importantly, a complex integrable system naturally emerges within this setup.

We define such a fibration by

$$\tilde{\mathcal{B}} = \Gamma \otimes \mathbb{R}/\mathbb{Z} \to \mathcal{B}, \tag{3.26}$$

with the lattice Γ we have defined. The fibers of $\tilde{\mathcal{B}}$ are (except at the singularities) $2r$-tori. Here we denote the corresponding fiber at u by X_u. $a(u)$ and $a_D(u)$ are solved as

$$a_I = \frac{1}{2\pi i}\oint_{A_I}\lambda,\ a_{DI} = \frac{1}{2\pi i}\oint_{B_I}\lambda, \tag{3.27}$$

where here A_I and B_I are homology cycles of the tori, and here λ is a meromorphic differential one-form on X_u, varying holomorphically with u. By λ, one can define a closed two form

$$\Omega = \mathrm{d}\lambda. \tag{3.28}$$

Note that Ω is gauge-invariant. We then have

$$\mathrm{d}a_I = \frac{1}{2\pi i}\oint_{A_I}\Omega,\ \mathrm{d}a_{DI} = \frac{1}{2\pi i}\oint_{B_I}\Omega. \tag{3.29}$$

In this context, the metric $\mathrm{Im}\,\tau$ defined by (3.22) is positive definite *if and only if* Ω is non-degenerate [52], i.e., Ω_{ij} $(i, j = 1 \cdots 2r)$ is invertible in any local coordinate system on X. (Here we denote ins inverse matrix as Ω^{ij}.) More concretely, the non-degeneracy means that locally Ω can take the form

$$\Omega = \sum_I \mathrm{d}x_I \wedge \mathrm{d}u^I, \tag{3.30}$$

where u_I are coordinates on $\mathcal{B}$ and x^I are some "conjugate" coordinates along the fibers. One then can define the Poisson bracket for two holomorphic functions f and g by

$$\{f, g\} \equiv \sum_{i,j} \Omega^{ij} \partial_i f \partial_j g. \tag{3.31}$$

(Note that the Poisson bracket obeys the Jacobi identity by the closure of Ω: $\mathrm{d}\Omega = 0$.) Then, according to (3.30),

$$\{u_I, u_I\} = 0, \ \{x_I, x_I\} = 0, \ \{u^I, x_I\} = \delta^I_J, \tag{3.32}$$

indicating that the u_i are a maximal set of commuting Hamiltonians. Therefore, the fibration $\tilde{\mathcal{B}} \to \mathcal{B}$ describes an *integrable system.*

To formulate the discussion in a more concrete manner, one can chose a local symplectic basis $\{\alpha_I, \beta^I\}$ of Γ, and write a point ϑ along the fiber $\tilde{\mathcal{B}}_u$ as

$$\vartheta = \vartheta^I_m \alpha_I + \vartheta_{e,I} \beta^I. \tag{3.33}$$

Then $(\vartheta^I_m, \vartheta_{e,I})$ are periodic coordinates on $\tilde{\mathcal{B}}_u$. Given this, one can define the meromorphic differential one-form λ by

$$\lambda = Z(u) \cdot \mathrm{d}\vartheta = a^I \mathrm{d}\vartheta_{e,I} + a_{D,I} \mathrm{d}\vartheta^I_m, \tag{3.34}$$

where the central charge $Z(u)$ is defined by (3.18). Thus, the holomorphic two form reads

$$\Omega = \mathrm{d}Z \cdot \mathrm{d}\vartheta = \mathrm{d}a^I \wedge \mathrm{d}\omega_I, \tag{3.35}$$

where ω_I are the complex coordinates on $\tilde{\mathcal{B}}_u$ defined as

$$\omega_I = \vartheta_{e,I} + \tau_{IJ}\vartheta_m^J, \tag{3.36}$$

and where we have used the relation $\partial\tau_{IJ}/\partial a^K = \partial\tau_{KJ}/\partial a^I$.

As shown by (3.28) and (3.30), Ω is closed and nondegenerate, so it is a holomorphic symplectic form on $\tilde{\mathcal{B}}$. The fibers of $\tilde{\mathcal{B}}$ are Lagrangian subvarieties with respect to Ω. The associated Poisson brackets are

$$\{a^I, a^J\} = \{\omega_I, \omega_J\} = 0, \quad \{a^I, \omega_J\} = \delta^I_J. \tag{3.37}$$

There are r independent Poisson-commuting complex quantities a^I in the phase space $\tilde{\mathcal{B}}$ of complex dimension $2r$. Hence, the fibration $\tilde{\mathcal{B}} \to \mathcal{B}$ describes an integrable system in the complex sense. Thus, we have seen that Seiberg-Witten theory really corresponds to the complex integrable system.

3.1.3 Emergence of Integrable System via Compactification to Three Dimensions

In the previous section we depicted how to encode the low-energy physics of four-dimensional $\mathcal{N} = 2$ gauge field theory in the complex integrable system. One may wonder whether the complex integrable system could emerge in a more physical way. The answer is yes: it emerges as the target space of a sigma model that arises when the theory is compactified on a circle [54]. Let us reveal this emergence in this section. The physical picture depicted in this section will guide us to find the correspondence between a four-dimensional $\mathcal{N} = 2$ supersymmetric gauge theory in the low energy limit and a quantum integrable system using our setup, as we shall see in section 2.

First, starting from a four-dimensional $\mathcal{N} = 2$ supersymmetric gauge theory, we compactify the x^4-direction to a circle S^1 of radius R. We take $R \gg 1/\Lambda$. Then, the dynamics at low energies $\mu \ll \Lambda$ but still $\mu \gg 1/R$ is described by essentially the same effective theory as the Seiberg-Witten theory we considered previously. The only difference is that this time the effective theory is formulated on $\mathbb{R}^3 \times S^1$ rather that $\mathbb{R}^4$. Due to this difference, compared with Seiberg-Witten theory, here the theory would receive finite-size corrections to the prepotential $\mathcal{F}$, which vanish in the limit $R \to \infty$. Further in the infrared, at energies $\mu \ll 1/R$, the theory is effectively a three-dimensional theory, which is what we are interested in.

How do we obtain such a three-dimensional theory? One may naturally think of doing the simple dimensional reduction of the low-energy effective theory on the S^1. However, it is not so simple, even though here at energies $\mu \ll 1/R$ the Kaluza-Klein modes are very massive and decouple. This is because the effective theory on $\mathbb{R}^3 \times S^1$ supports topologically nontrivial configurations in which the worldlines of BPS particles wrap the S^1. Such configurations appear as instantons in three dimensions; and the action for these instantons is roughly $2\pi R|Z|$ (where Z is the central charge), which is not necessarily large and may not decouple.

Nonetheless, if R is very large, the instanton effects are suppressed. Thus, for sufficiently large R (note that $\mu \ll 1/R$ holds), the effective three-dimensional Lagrangian is obtained by dimensional reduction of the four-dimensional Lagrangian (whose bosonic part takes the form of (3.25)), as long as one stays away from the singular loci in the moduli space $\mathcal{B}$ where some BPS particles become massless. Let us consider this case and identify the corresponding three-dimensional theory.

Dimensional reduction for the scalars a^I is straightforward (note that at the low energy limit we are considering, a^I is independent of x^4). For the gauge field, we note that at each point on the $\mathbb{R}^3$, the components A_4^I describe connections on line bundles over the S^1. Since gauge connections on S^1 are determined up to gauge transformations by their holonomies,

$$\exp\left(i \oint A_4^I \, \mathrm{d}x^4\right), \tag{3.38}$$

where the x^4-dependence is integrated out, modulo gauge transformations, the gauge connections on S^1 can be equivalently defined as

$$\oint A_4^I \mathrm{d}x^4 = \oint \frac{\theta_e^I}{2\pi R} = \theta_e^I, \tag{3.39}$$

where θ_e^I are periodic scalars with periodicity 2π and are independent of x^4. The residual gauge symmetry is given by the gauge transformations on the $\mathbb{R}^3$. Plugging the expression (3.39) into the effective Lagrangian (3.25), dropping all the x^4-dependence and integrating over the x^4-direction, we get the three-dimensional Lagrangian

$$\mathcal{L}^{(3)} = \frac{R}{2}\,\mathrm{Im}\,\tau_{IJ}\left(\mathrm{d}a^I\wedge\star\mathrm{d}\bar{a}^J+F^{(3),I}\wedge\star F^{(3),J}+\frac{\mathrm{d}\theta_e^I\wedge\star\mathrm{d}\theta_e^J}{4\pi^2R^2}\right)+\frac{i}{2\pi}\,\mathrm{Re}\,\tau_{IJ}F^{(3),I}\wedge\mathrm{d}\theta_e^J. \tag{3.40}$$

Here $F^{(3),I}$ are the field strengths of the gauge fields $A^{(3),I}$, coming from the remaining components of A^I.

In order to obtain the sigma model in three dimensions (whose target space will give us the desired integrable system), we need to dualize the gauge fields to scalars. To do this, we convert the path integral variables from $A^{(3),I}$ to $F^{(3),I}$. The constraint $F^{(3),I}$ must obey is that through any closed surface $S\subset\mathbb{R}^3$, their magnetic fluxes must be integers:

$$\frac{1}{2\pi}\int_S F^{(3),I}\in\mathbb{Z}. \tag{3.41}$$

(If $A^{(3),I}$ are connections on line bundles L_I, then $F^{(3),I}/2\pi$ represent the first Chern classes $c_1(L_I)\in H^1(S;\mathbb{Z})$.) To realize such a constraint, we introduce periodic scalars $\theta_{m,I}$ of periodicity 2π as Lagrange multipliers, and add to the action the term

$$-\frac{i}{2\pi}\int_{\mathbb{R}^3}F^{(3),I}\wedge\mathrm{d}\theta_{m,I}. \tag{3.42}$$

Integrating $\theta_{m,I}$ out produces the constraint 3.41, as explained in the following. Consider a continuous configuration such that $\theta_{m,I}$ jumps by $2\pi n_I$ for some $n_I\in\mathbb{Z}$ as we cross S from inside. Then $\mathrm{d}\theta_{m,I}$ contains $2\pi n_I\delta(S)$, where $\delta(S)$ is a two-form with delta-function support on S which represents the Poincaré dual of the homology class $[S]$. Thus the added term contains the factor

$$-in_I\int_S F^{(3),I}, \tag{3.43}$$

and a summation over n_I produces the desired constraint. Given this, we are now ready to carry out the dualization.

Integrating out $F^{(3),I}$ instead of $\theta_{m,I}$, we get the dualized Lagrangian

$$\mathcal{L}_D^{(3)} = \frac{R}{2}\,\mathrm{Im}\,\tau_{IJ}\big(\mathrm{d}a^I \wedge \star\mathrm{d}\bar{a}^J + \eta^I \wedge \star\bar{\eta}^J\big), \tag{3.44}$$

with

$$\eta^I = \frac{1}{2\pi R}(\mathrm{Im}\,\tau)^{-1,IJ}\big(\mathrm{d}\theta_{m,J} - \tau_{JK}\,\mathrm{d}\theta_e^K\big). \tag{3.45}$$

This is the bosonic Lagrangian for a sigma model with target space metric

$$g^{\mathrm{sf}} = R\,\mathrm{Im}\,\tau_{IJ}\left(\mathrm{d}a^I\,\mathrm{d}\bar{a}^J + \eta^I\,\bar{\eta}^J\right). \tag{3.46}$$

This "semiflat" metric g^{sf} is singular over the singular loci in $\mathcal{B}$, around which a^I have monodromies. Instantons correct g^{sf} to a smooth metric g.

The theory has $\mathcal{N} = 4$ supersymmetry in three dimensions, requiring the target space of the sigma model to be a hyperkähler manifold, which we denote as $\mathcal{M}$. This means that $\mathcal{M}$ has three independent complex structures J_α, $\alpha = 1$, 2, 3, obeying the relation

$$J_\alpha^2 = J_1 J_2 J_3 = -1, \tag{3.47}$$

and the metric g is Kähler with respect to each J_α. In the semiflat approximation, we can take J_α to act on $T^*\mathcal{M}$ as follows:

$$\begin{aligned}
J_1 &\colon (\mathrm{d}a^I, \eta^I) \mapsto (i\bar{\eta}^I, -i\mathrm{d}\bar{a}^I),\\
J_2 &\colon (\mathrm{d}a^I, \eta^I) \mapsto (-\bar{\eta}^I, \mathrm{d}\bar{a}^I),\\
J_3 &\colon (\mathrm{d}a^I, \eta^I) \mapsto (i\mathrm{d}a^I, i\eta^I).
\end{aligned} \tag{3.48}$$

The presence of these three complex structures induce three Kähler 2-forms ω_i $(i = 1, 2, 3)$ on $\mathcal{M}$, namely

$$\omega_i(X, Y) = g(J_i X, Y), \tag{3.49}$$

where X, Y are vectors in the tangent space of $\mathcal{M}$.

One can check that the semiflat metric (3.46) is indeed Kähler with respect to each of these complex structures. Identifying the exact hyperkähler structure of $\mathcal{M}$ is a difficult problem, and is closely related to the wall-crossing phenomenon of BPS spectrum [55].

With this in hand, now we turn to explain how the integrable system emerges. To this end, the first step is to understand the geometry of the target space $\mathcal{M}$ better, where $\mathcal{M}$ is a fibration over $\mathcal{B}$ whose fibers are $2r$-tori parametrized by the periodic scalars $(\theta_e^I, \theta_{m,I})$. In order to do so, we should go back to the four-dimensional description. In four dimensions we have the formula

$$\theta_e^I = \oint_C A^I, \tag{3.50}$$

where C is a cycle located at a point in $\mathbb{R}^3$ and wrapped on the S^1. On the other hand, the dualization procedure (i.e. integrating out $F^{(3),I}$) in three dimensions sets

$$\mathrm{d}\theta_{m,I} = \operatorname{Re}\tau_{IJ}\,\mathrm{d}\theta_e^J - 2\pi i R \operatorname{Im}\tau_{IJ} \star F^{(3),J}. \tag{3.51}$$

We can define

$$F_{D,I} = \operatorname{Re}\tau_{IJ}\,F^J - i\operatorname{Im}\tau_{IJ}\star F^J. \tag{3.52}$$

The equations of motion imply $\mathrm{d}F_{D,I} = 0$, so we can write

$$\theta_{m,I} = \oint_C A_{D,I}, \tag{3.53}$$

using gauge fields $A_{D,I}$ for $F_{D,I}$. Then as is clear from the symmetry between the equations $\mathrm{d}F^I = 0$ and $\mathrm{d}F_{D,I} = 0$, the field strengths F^I and $F_{D,I}$ are dual to each other, and together form a Γ^*-valued two-form $\mathbb{F} = F^I\beta_I + F_{D,I}\alpha^I$. (Here Γ_u^* denotes the dual lattice of Γ_u defined in (3.15), in our case $\Gamma^* = \Gamma$.) Similarly, the gauge fields A^I and $A_{D,I}$ form a Γ^*-valued gauge field $\mathbb{A} = A^I\beta_I + A_{D,I}\alpha^I$. So writing

$$\theta = \theta_e^I\beta_I + \theta_{m,I}\alpha^I, \tag{3.54}$$

we can combine the two formulas (3.50) and (3.53) into a single formula which is independent of the choice of symplectic basis:

$$\theta = \oint_C \mathbb{A}. \tag{3.55}$$

Thus θ is a map to $\Gamma_a^* \otimes \mathbb{R}/2\pi\mathbb{Z}$, while the a^I give a map $a\colon \mathbb{R}^3 \to \mathcal{B}$.

This consideration suggests $\mathcal{M} \cong \Gamma^* \otimes \mathbb{R}/2\pi\mathbb{Z}$. And since $\Gamma^* = \Gamma$, this space is isomorphic to the Seiberg-Witten fibration $\tilde{\mathcal{B}} = \Gamma \otimes \mathbb{R}/\mathbb{Z}$:

$$\mathcal{M} \cong \tilde{\mathcal{B}}. \tag{3.56}$$

If we identify $\theta = 2\pi\vartheta$ under this isomorphism, then we have the relations

$$\theta_e^I = -2\pi\vartheta_m^I, \qquad \theta_{m,I} = 2\pi\vartheta_{e,I}. \tag{3.57}$$

The holomorphic symplectic form Ω is identified as

$$\Omega = \frac{1}{2\pi}\mathrm{d}a^I \wedge \mathrm{d}z_I = -i(\omega_1 + i\omega_2), \tag{3.58}$$

where we equipped the fibers with complex coordinates

$$z_I = \theta_{m,I} - \tau_{IJ}\theta_e^J = 2\pi w_I. \tag{3.59}$$

Here respectively, a^I and z_I are r independent Poisson-commuting complex quantities. Thus finally, we have succeeded in obtaining the desired complex integrable system.

However, here we should treat the discussion more carefully. Actually, it is not entirely true that $\mathcal{M}$ is isomorphic to $\Gamma^* \otimes \mathbb{R}/2\pi\mathbb{Z}$. The reason is that whereas θ_e^I are determined by the formula (3.50), the relation (3.51) determines the corresponding formula (3.53) only up to a constant. Thus we have a collection of constants, each associated to an open patch in $\mathcal{B}$ equipped with a chosen symplectic basis. Locally we can discard these constants since the Lagrangian depends on $\theta_{m,I}$ only through their derivatives. Globally, setting all of them to zero consistently may not be possible. Indeed, it was observed in [55] that $\theta_{m,I}$ can have monodromy shifting them by π. Nonetheless, since such monodromy leaves the holomorphic symplectic form invariant, it *does not* affect the fact that the fibration $\mathcal{M} \to \mathcal{B}$ defines an integrable system,

Lastly, we should clarify what happens to the integrable system structure when the instanton corrections are included. The structure is associated with the complex structure J_3, which is special among all the complex structures of $\mathcal{M}$

in the sense that it is the only complex structure under which Z is holomorphic. Instanton corrections are accompanied with a factor of $\exp(-2\pi R|Z|)$, so cannot arise in quantities that are holomorphic in J_3. This implies that J_3 itself and the associated holomorphic two-form Ω, and hence also the integrable system structure, are *protected* against the instanton corrections.

3.1.4 From Classical to Quantum Integrable System

So far, we have seen that $\mathcal{N} = 2$ supersymmetric gauge theories in four dimensions at low energies can be related to *classical* complex integrable systems. One may naturally wonder whether this connection can be "quantized", that is, whether any connection can be established between the $\mathcal{N} = 2$ theories and *quantum* integrable systems. A few years ago, Nekrasov and Shatashvili [23] answered this question with a resounding *yes*: they found that the Omega-deformation of $\mathcal{N} = 2$ supersymmetric gauge theories correspond to a *quantized* complex integrable system. Let us briefly review their result.

For simplicity, we consider the Omega-deformed theory in two dimensions. The effective superpotential $W^{eff}(a)$ is defined by

$$W^{eff}(a) = \frac{\mathcal{F}(a)}{\varepsilon}, \tag{3.60}$$

where ε is the Omega-deformation parameter.

Consequently, the meromorphic differential one-form is defined by

$$\lambda = \frac{a^I}{\varepsilon}\mathrm{d}\vartheta_{e,I} + \frac{a_{D,I}}{\varepsilon}\mathrm{d}\vartheta_m^I, \tag{3.61}$$

where we have the same fibration of (3.26) and use the same coordinates of (3.33) along the fibers.

This leads to the holomorphic two-form reading

$$\Omega = \frac{\mathrm{d}a}{\varepsilon} \wedge \mathrm{d}\omega, \tag{3.62}$$

where

$$\omega_I = \vartheta_{e,I} + \tau_{IJ}\vartheta_m^J. \tag{3.63}$$

The quantization condition is given by

$$\frac{\partial W^{eff}}{\partial a^I} = \frac{\partial \mathcal{F}}{\varepsilon \partial a^I} = n \tag{3.64}$$

with n an integer. Thus the integrable system is quantized, with the parameter ε playing the role of Planck constant.

In the next section we shall establish another, yet closely related, connection between four-dimensional $\mathcal{N} = 2$ supersymmetric gauge theories and quantum integrable systems. Instead of turning on the Omega-deformation, our setup is an $\mathcal{N} = 2$ supersymmetric gauge theory formulated on $S^2 \times \mathbb{R} \times S^1$, which is topologically twisted along $\mathbb{R} \times S^1$. We will show that the low-energy dynamics of this theory is described by a quantum integrable system, with the Planck constant set by $1/2r$. This system quantizes the real integrable system whose symplectic form is $\mathrm{Re}\,\Omega$, where Ω is the holomorphic symplectic form of the complex integrable system associated to the Coulomb branch.

3.2 Effective Theory of the $\mathcal{N} = 2$ Theory on $S^2 \times \mathbb{R} \times S^1$

In the first part of this section, we formulate an $\mathcal{N} = 2$ supersymmetric gauge theory on $S^2 \times \mathbb{R} \times S^1$, which is topologically twisted along $\mathbb{R} \times S^1$. The three-dimensional effective theory can be obtained by compactifying the four-dimensional effective theory. Then the dualization that we introduced in the previous section will turn the three-dimensional effective theory into a sigma model on $S^2 \times \mathbb{R}$. Then, as we shall see, if we localize the the sigma model on S^2, we can obtain a quantum mechanical system which is the desired quantum integrable system.

However, by the properties of the supercharges of the low-energy theory, in contrast to direct dimensional reduction, we can take a different strategy to construct the sigma model. Using such a strategy to construct the sigma model will make up the second part of the section.

3.2.1 The $\mathcal{N}=2$ Supersymmetric Gauge Theory on $S^2 \times \mathbb{R} \times S^1$

Here we would like to formulate the $\mathcal{N}=2$ supersymmetric gauge theory on $S^2 \times \mathbb{R} \times S^1$. To this end, we first treat a more general setup: formulating an $\mathcal{N}=2$ supersymmetric gauge theory on $S^2 \times C$, where the cylinder $\mathbb{R} \times S^1$ is replaced with an arbitrary Riemann surface C.

For a general choice of C equipped with a curved metric, C admits no covariantly constant spinors (which are parameters of the supersymmetry transformation), which leads to supersymmetry being completely broken. In order to preserve some supersymmetry, we must topologically twist the theory along C. The twist is done as follows.

On $S^2 \times C$, the structure group of the spin connection reduces to $U(1)_{S^2} \times U(1)_C$, under which, the supercharges transform as

$$(\pm 1, \pm 1, \pm 1). \tag{3.65}$$

We replace $U(1)_C$ by the diagonal subgroup $U(1)'_C = U(1)_C \times U(1)_R$. Under $U(1)'_C$, the supercharges transform as

$$(\pm 1, 0, \pm 1) \oplus (\pm 1, 2, 1) \oplus (\pm 1, -2, -1), \tag{3.66}$$

which shows that four of them are now scalars. The four scalar supercharges can thus be preserved on C.

At the same time, on S^2, the four supercharges which are scalars on C are still spinors. Due to the symmetric nature of its geometry, they can all be preserved on S^2. It turns out that two of the four supercharges have positive chirality while the other two have negative chirality on S^2. Thus after the twisting, we get $\mathcal{N}=(2,2)$ supersymmetry [58, 59] on S^2. The associated transformation parameters are conformal Killing spinors ε, $\bar{\varepsilon}$, obeying the Killing equations:

$$\nabla_\mu \varepsilon = +\frac{1}{2r}\gamma_\mu \gamma_{\hat{3}} \varepsilon, \qquad \nabla_\mu \bar{\varepsilon} = -\frac{1}{2r}\gamma_\mu \gamma_{\hat{3}} \bar{\varepsilon}, \tag{3.67}$$

where $\mu = 1, 2$ is the coordinate index on S^2.[1] Each of these equations has two independent solutions, so in total we have four, ε_α, $\bar{\varepsilon}_\alpha$, $\alpha = 1, 2$. We write $\bar{Q}_\alpha$, Q_α for the supercharges corresponding to ε_α, $\bar{\varepsilon}_\alpha$, and $\bar{\mathcal{Q}}_\alpha$, $\mathcal{Q}_\alpha$ for their action on fields, respectively.

In addition to the four supersymmetries generated by $\bar{Q}_\alpha$, Q_α, the $\mathcal{N} = (2,2)$ supersymmetry group contains the rotations of the S^2 and a $U(1)$ R-symmetry. We choose the R-symmetry to be the vector R-symmetry $U(1)_V$. (So we are considering A-type supersymmetry [71].) The R-symmetry rotates $\bar{Q}_\alpha$ by charge $q = +1$ and Q_α by $q = -1$. The nonvanishing commutators among the supercharges are

$$\{\bar{Q}_\alpha, Q_\beta\} = \mathcal{L}_\xi + i\alpha F_V \tag{3.68}$$

modulo gauge transformations. On the right-hand side appear the Lie derivative $\mathcal{L}_\xi$ by the Killing vector field $\xi^\mu = i\varepsilon_\alpha\gamma^\mu\bar{\varepsilon}_\beta$, as well as the $U(1)_V$ generator F_V accompanied with the parameter $\alpha = \varepsilon_\alpha\gamma_{\hat{3}}\bar{\varepsilon}_\beta/2r$. Note that the commutators *do not* generate translations along C, since our supercharges are scalars on C. As a result, the commutation relations remain unchanged from the two-dimensional case, even though we are really dealing with a four-dimensional theory on $S^2 \times C$.

To construct the four-dimensional theory, we would like to repackage the field content of the twisted theory into supermultiplets of $\mathcal{N} = (2,2)$ supersymmetry. In general $U(1)_R$ is the only $U(1)$ R-symmetry present in the twisted theory, so this identified with $U(1)_V$. (There is another $U(1)$ R-symmetry if the theory is superconformal.) The fact that the vector multiplet scalar is neutral under $U(1)_R$ means that the theory should be formulated using vector and chiral multiplets, as opposed to twisted vector and chiral multiplets. (Unlike the case of flat spacetime, twisted and untwisted multiplets are inequivalent representations on S^2.)

Next, we would like to write down the supersymmetry transformation rules and supersymmetric Lagrangians for the theory. As explained, the theory is

[1] Our conventions for spinors on S^2 are as follows. We use spherical coordinates $(x^1, x^2) = (\theta, \varphi)$ on S^2 such that the round metric of radius r is $r^2 \mathrm{d}\theta^2 + r^2 \sin^2\theta \, \mathrm{d}\varphi^2$. The hatted index $\hat{\mu} = \hat{1}, \hat{2}$ refers to the orthonormal frame $e_{\hat{1}} = \partial_1/r$, $e_{\hat{2}} = \partial_2/r\sin\theta$. Often we extend $\hat{\mu}$ to run from $\hat{1}$ to $\hat{3}$. The gamma matrices $\gamma_{\hat{\mu}}$ are given by the Pauli matrices, and the chirality operator is $\gamma_{\hat{3}}$. The product of Dirac spinors $\psi\chi = \psi^T C\chi$, with $C = i\gamma_{\hat{2}}$. The spin connection is denoted by ∇.

twisted along C and possesses $\mathcal{N} = (2,2)$ supersymmetry on S^2, so we are going to construct the supersymmetry transformation rules and supersymmetric Lagrangians of the 4d theory by "lifting" their counterparts of the $\mathcal{N} = (2,2)$ supersymmetric gauge theory on S^2 [58, 59] (the meaning of "lifting" will be clear soon). In order to do so, first of all, we need to understand how the vector multiplet and hypermultiplet of the $\mathcal{N} = 2$ four-dimensional theory decompose as supermultiplets of $\mathcal{N} = (2,2)$ supersymmetry in 2d.

For the vector multiplet, after the twisting, four components λ, $\bar{\lambda}$ of the gauginos become scalars on C and Dirac spinors on S^2. Together with the vector multiplet scalar $\phi = \phi_1 + i\phi_2$, the components A_μ ($\mu = 1, 2,$) of the gauge field along S^2, and a real auxiliary field D, they form an $\mathcal{N} = (2,2)$ vector multiplet V:

$$V = (\phi, \lambda, \bar{\lambda}, A_\mu, D). \tag{3.69}$$

The rest of the $\mathcal{N} = 2$ vector multiplet fields are divided into two groups according to their transformation properties under $U(1)_C$. We choose a holomorphic coordinate z on C such that $(1,0)$-forms have charge -2. Then, one group form an $\mathcal{N} = (2,2)$ chiral multiplet Φ_z of R-charge $q = 0$ in the adjoint representation together with a complex auxiliary field F_z, while the other form the corresponding antichiral multiplet $\bar{\Phi}_{\bar{z}}$:

$$\Phi_z = (A_z, \lambda_z, F_z), \qquad \bar{\Phi}_{\bar{z}} = (A_{\bar{z}}, \bar{\lambda}_{\bar{z}}, \bar{F}_{\bar{z}}). \tag{3.70}$$

Our convention for chiral multiplets is that if the scalar component has R-charge q, then the spinor has R-charge $q - 1$.

Now we turn to hypermultiplets. A hypermultiplet of the 4d theory consists of two $\mathcal{N} = 1$ chiral multiplets. If we write M and $\tilde{M}^\dagger$ for the scalars of these chiral multiplets and assign them R-charge $q = +1$ and -1, then after the twisting they become sections M_+ and $\tilde{M}_-^\dagger$ of $\bar{K}_C^{1/2}$ and $K_C^{1/2}$, respectively. These are part of a chiral multiplet H_+ and an antichiral multiplet $\tilde{H}_-^\dagger$ of $\mathcal{N} = (2,2)$, both in the same representation of the 4d hypermultiplet:

$$H_+ = (M_+, \psi_+, F_+), \qquad \tilde{H}_-^\dagger = (\tilde{M}_-^\dagger, \tilde{\psi}_-^\dagger, \tilde{F}_-^\dagger). \tag{3.71}$$

Their hermitian conjugates are part of an antichiral multiplet $H_-^\dagger$ and a chiral multiplet H_+ in the dual representation:

$$H_-^\dagger = (M_-^\dagger, \psi_-^\dagger, F_-^\dagger), \qquad \tilde{H}_+ = (\tilde{M}_+, \tilde{\psi}_+, \tilde{F}_+). \tag{3.72}$$

The supersymmetry transformation rules for these multiplets are as follows: for V,[2]

$$\begin{aligned}
\delta A_\mu &= -\frac{i}{2}\big(\bar{\varepsilon}\gamma_\mu\lambda + \varepsilon\gamma_\mu\bar{\lambda}\big), \\
\delta\phi &= \bar{\varepsilon}\gamma_-\lambda - \varepsilon\gamma_+\bar{\lambda}, \\
\delta\bar{\phi} &= \bar{\varepsilon}\gamma_+\lambda - \varepsilon\gamma_-\bar{\lambda}, \\
\delta\lambda &= i\Big[\Big(F_{\hat{1}\hat{2}} + \frac{\phi_1}{r}\Big)\gamma_{\hat{3}} + \gamma_-\not{D}\phi + \gamma_+\not{D}\bar{\phi} + \frac{1}{2}[\phi,\bar{\phi}]\gamma_{\hat{3}} + iD\Big]\varepsilon, \\
\delta\bar{\lambda} &= i\Big[\Big(F_{\hat{1}\hat{2}} + \frac{\phi_1}{r}\Big)\gamma_{\hat{3}} - \gamma_+\not{D}\phi - \gamma_-\not{D}\bar{\phi} - \frac{1}{2}[\phi,\bar{\phi}]\gamma_{\hat{3}} - iD\Big]\bar{\varepsilon}, \\
\delta D &= -\frac{i}{2}\bar{\varepsilon}\big(\not{D}\lambda + [\phi,\gamma_+\lambda] + [\bar{\phi},\gamma_-\lambda]\big) + \frac{i}{2}\varepsilon\big(\not{D}\bar{\lambda} - [\phi,\gamma_-\bar{\lambda}] - [\bar{\phi},\gamma_+\bar{\lambda}]\big);
\end{aligned} \tag{3.73}$$

for Φ_z, $\bar{\Phi}_{\bar{z}}$,

$$\begin{aligned}
\delta A_z &= \bar{\varepsilon}\lambda_z, \\
\delta A_{\bar{z}} &= \varepsilon\bar{\lambda}_{\bar{z}}, \\
\delta\lambda_z &= \big(i\gamma^\mu F_{\mu z} + D_z\phi\gamma_+ + D_z\bar{\phi}\gamma_-\big)\varepsilon + F_z\bar{\varepsilon}, \\
\delta\bar{\lambda}_{\bar{z}} &= \big(i\gamma^\mu F_{\mu\bar{z}} - D_{\bar{z}}\phi\gamma_- - D_{\bar{z}}\bar{\phi}\gamma_+\big)\bar{\varepsilon} + \bar{F}_{\bar{z}}\varepsilon, \\
\delta F_z &= i\varepsilon\big(\not{D}\lambda_z - \gamma_-[\phi,\lambda_z] - \gamma_+[\bar{\phi},\lambda_z] + iD_z\lambda\big), \\
\delta\bar{F}_{\bar{z}} &= i\bar{\varepsilon}\big(\not{D}\bar{\lambda}_{\bar{z}} - \gamma_+[\bar{\lambda}_{\bar{z}},\phi] - \gamma_-[\bar{\lambda}_{\bar{z}},\bar{\phi}] + iD_{\bar{z}}\bar{\lambda}\big);
\end{aligned} \tag{3.74}$$

and for H_+, $H_-^\dagger$,

$$\begin{aligned}
\delta M_+ &= \bar{\varepsilon}\psi_+, \\
\delta M_-^\dagger &= \varepsilon\psi_-^\dagger, \\
\delta\psi_+ &= i\Big(\not{D}M_+ + \phi M_+\gamma_+ + \bar{\phi}M_+\gamma_- + \frac{1}{2r}M_+\gamma_{\hat{3}}\Big)\varepsilon + F_+\bar{\varepsilon}, \\
\delta\psi_-^\dagger &= i\Big(\not{D}M_-^\dagger + M_-^\dagger\phi\gamma_- + M_-^\dagger\bar{\phi}\gamma_+ - \frac{1}{2r}M_-^\dagger\gamma_{\hat{3}}\Big)\bar{\varepsilon} + F_-^\dagger\varepsilon, \\
\delta F_+ &= i\varepsilon\Big(\not{D}\psi_+ - \gamma_-\phi\psi_+ - \gamma_+\bar{\phi}\psi_+ - \lambda M_+ + \frac{1}{2r}\gamma_{\hat{3}}\psi_+\Big), \\
\delta F_-^\dagger &= i\bar{\varepsilon}\Big(\not{D}\psi_-^\dagger - \gamma_+\psi_-^\dagger\phi - \gamma_-\psi_-^\dagger\bar{\phi} + M_-^\dagger\bar{\lambda} - \frac{1}{2r}\gamma_{\hat{3}}\psi_-^\dagger\Big).
\end{aligned} \tag{3.75}$$

[2] Our definition of D differs from that in [58] by the shift $D \to D + \phi_2/r$.

The supersymmetry transformations for $\tilde{H}_+$, $\tilde{H}_-^\dagger$ are obtained from those for H_+, $H_-^\dagger$ by replacing the fields appropriately. In the above formulas, $\gamma_\pm = (1 \pm \gamma_{\hat{3}})/2$ are the projectors to the positive and negative chirality subspaces, and $\not{D} = \gamma^\mu D_\mu$ with $D = \nabla - iA$ the covariant derivative coupled to the spin connection and the gauge field.

The standard supersymmetric Lagrangian on S^2 for vector and chiral multiplets lift to the following Lagrangians for V and Φ_z, $\bar{\Phi}_{\bar{z}}$:

$$\begin{aligned}\mathcal{L}_V = \frac{1}{2}\operatorname{Tr}\biggl[\Bigl(F_{\hat{1}\hat{2}} + \frac{\phi_1}{r}\Bigr)^2 + D^\mu\phi D_\mu\bar{\phi} + \frac{1}{4}[\phi,\bar{\phi}]^2 + D^2 \\ + i\lambda\bigl(\not{D}\lambda + [\phi,\gamma_+\lambda] + [\bar{\phi},\gamma_-\lambda]\bigr)\biggr],\end{aligned} \tag{3.76}$$

$$\begin{aligned}\mathcal{L}_\Phi = \operatorname{Tr}\biggl[F^{\mu z}F_{\mu z} + \frac{1}{2}\bigl(D^z\phi D_z\bar{\phi} + D^{\bar{z}}\phi D_{\bar{z}}\bar{\phi}\bigr) + \Bigl(D + \frac{\phi_2}{r}\Bigr)F^z{}_z + \bar{F}^z F_z \\ - i\bar{\lambda}^z\bigl(\not{D}\lambda_z - [\phi,\gamma_-\lambda_z] - [\bar{\phi},\gamma_+\lambda_z]\bigr) + \bar{\lambda}^z D_z\lambda + D^z\bar{\lambda}\lambda_z\biggr].\end{aligned} \tag{3.77}$$

The $\mathcal{N} = 2$ vector multiplet action on $S^2 \times C$ is simply

$$\frac{1}{e^2}\int_{S^2\times C}\mathrm{vol}_{S^2\times C}(\mathcal{L}_V + \mathcal{L}_\Phi) + \frac{i\theta}{8\pi^2}\int_{S^2\times C} F\wedge F. \tag{3.78}$$

Here $\mathrm{vol}_{S^2\times C}$ is the volume form of $S^2 \times C$. We see that the action contains all the required kinetic terms. In particular, the $F^{z\bar{z}}F_{z\bar{z}}$ term arises from integrating out the auxiliary field D.

For the hypermultiplet, the Lagrangian for H_+, $H_-^\dagger$ obtained from the corresponding chiral multiplet Lagrangian in two dimensions is

$$\begin{aligned}\mathcal{L}_H = \sqrt{h^{z\bar{z}}}\biggl[D^\mu M_-^\dagger D_\mu M_+ + M_-^\dagger\Bigl(\frac{1}{2}\{\phi,\bar{\phi}\} + iD + \frac{1}{4r^2}\Bigr)M_+ + F_-^\dagger F_+ \\ - i\psi_-^\dagger\Bigl(\not{D} - \phi\gamma_- - \bar{\phi}\gamma_+ + \frac{1}{2r}\gamma_{\hat{3}}\Bigr)\psi_+ + i\psi_-^\dagger\lambda M_+ - iM_-^\dagger\bar{\lambda}\psi_+\biggr].\end{aligned} \tag{3.79}$$

The Lagrangian $\mathcal{L}_{\tilde{H}}$ for $\tilde{H}_+$, $\tilde{H}_-^\dagger$ is similar. To get the kinetic terms along C, we must turn on a superpotential. Up to an overall phase, the right choice is

$$W = \sqrt{2}h^{z\bar{z}}\tilde{M}_+ D_z M_+. \tag{3.80}$$

This is part of a chiral multiplet whose auxiliary field

$$F_W = \sqrt{2}h^{z\bar{z}}\big(\tilde{F}_+ D_z M_+ - D_z\tilde{M}_+ F_+ - i\tilde{M}_+ F_z M_+ \\ - \tilde{\psi}_+ D_z\psi_+ + i\tilde{\psi}_+\lambda_z M_+ + i\tilde{M}_+\lambda_z\psi_+\big). \quad (3.81)$$

The complex conjugate $\bar{W}$ of W is part of an antichiral multiplet. If we write $\bar{F}_W$ for its auxiliary field, the F-term is given by

$$\mathcal{L}_W = i\big(F_W + \bar{F}_W\big). \quad (3.82)$$

The hypermultiplet action is then

$$\frac{1}{e^2}\int_{S^2\times C}\mathrm{vol}_{S^2\times C}\big(\mathcal{L}_H + \mathcal{L}_{\tilde{H}} + \mathcal{L}_W\big). \quad (3.83)$$

As usual, hypermultiplet masses can be introduced by weakly gauging flavor symmetries and giving vacuum expectation values to the vector multiplet scalars.

After integrating out the auxiliary fields, the bosonic part of the total Lagrangian becomes

$$\frac{1}{2}\operatorname{Tr}\bigg[\Big(F_{\hat{1}\hat{2}} + \frac{\phi_1}{r}\Big)^2 + 2F^{\mu z}F_{\mu z} + F^{z\bar{z}}F_{z\bar{z}} + D^m\phi D_m\bar{\phi} + \frac{1}{4}[\phi,\bar{\phi}]^2 + \frac{2}{r}\phi_2 F^z{}_z\bigg] \\ + \sqrt{h^{z\bar{z}}}\Big(D^m M_-^\dagger D_m M_+ + D^m\tilde{M}_+ D_m\tilde{M}_-^\dagger - M_-^\dagger R^z{}_z M_+ - \tilde{M}_+ R^z{}_z\tilde{M}_-^\dagger \\ + \frac{1}{4r^2}\big(M_-^\dagger M_+ + \tilde{M}_+\tilde{M}_-^\dagger\big) + \frac{1}{2}M_-^\dagger\{\phi,\bar{\phi}\}M_+ + \frac{1}{2}\tilde{M}_+\{\phi,\bar{\phi}\}\tilde{M}_-^\dagger\Big) \\ + \frac{1}{2}\big\|M_-^\dagger T_a M_+ - \tilde{M}_+ T_a\tilde{M}_-^\dagger\big\|^2 + 2\|\tilde{M}_+ T_a M_+\|^2, \quad (3.84)$$

where m runs from 1 to 4, $R^z{}_z = [\nabla^z, \nabla_z]$, T_a are generators of the gauge symmetry in the representation R, and the norm on the Lie algebra is given by the Killing form. If we drop the terms with explicit r dependence, this reproduces precisely the bosonic Lagrangian for the theory on $\mathbb{R}^4$. Therefore the above Lagrangian describes the theory formulated on $S^2 \times C$.

We remark that the Lagrangian (3.84) contains the mass terms for the hypermultiplet scalars with mass proportional to $1/r$. So they are set to zero in vacua; there is no Higgs branch.

The pieces $\mathcal{L}_V$, $\mathcal{L}_H$ and $\mathcal{L}_{\tilde{H}}$ of the total Lagrangian can be written in Q-exact forms for an appropriate choice of a supercharge Q. For example, we have

$$\mathcal{L}_V = \frac{1}{2}\mathcal{Q}\big[\mathcal{Q}_2 \operatorname{Tr}(\bar{\lambda}\bar{\lambda}) + \zeta^{-1}\bar{\mathcal{Q}}_1 \operatorname{Tr}(\lambda\lambda)\big], \tag{3.85}$$

$$\mathcal{L}_H = \frac{1}{2}\mathcal{Q}\big[\mathcal{Q}_2(F_-^\dagger M_+) + \zeta^{-1}\bar{\mathcal{Q}}_1(M_-^\dagger F_+)\big], \tag{3.86}$$

for any $Q = Q_1 + \zeta\bar{Q}_2$ with $\zeta \in \mathbb{C}^\times$. The other pieces $\mathcal{L}_\Phi$ and $\mathcal{L}_W$ are not Q-exact. (A formula similar to the one for $\mathcal{L}_H$ would not work for $\mathcal{L}_\Phi$, since the scalar A_z of Φ_z is not a globally-defined object.) Nevertheless, these terms do not introduce dependence on the Kähler structure of C, since the volume form of C is given by $\mathrm{vol}_C = ih_{z\bar{z}}\mathrm{d}z \wedge \mathrm{d}\bar{z}$ and $\mathrm{vol}_C\, h^{z\bar{z}}$ is independent of h. It follows that the twisted theory is independent of the Kähler structure if we regard Q as a BRST operator.

Since hypermultiplets are spinors on C after the twisting, formulating the twisted theory requires picking a spin structure on C. We can avoid this by redefinition of the $U(1)_R$ symmetry used in the twisting. The theory has a global symmetry $U(1)_B$ under which H and $\tilde{H}$ have opposite charges. We can shift $U(1)_R$ by $U(1)_B$ so that the hypermultiplets have integer R-charges, say $q = 2$ for H and $q = 0$ for $\tilde{H}$. Then the twisting turns H into a $(0,1)$-form and $\tilde{H}$ into a scalar on C. For this vector R-charge assignment,[3] there are no mass terms due to the curvature of S^2 and there can be a Higgs branch.

Importantly, there is one point deserved to be mentioned. The twisted theory here has four supercharges, and any of their linear combinations can be used as a BRST operator. For our purpose, we want the theory to be independent of the Kähler structure on C, which requires the linear combinations of the supercharges to be special. To this end, we choose the parameters in such a way that $\bar{\varepsilon}_\alpha = \gamma_{\hat{3}}\varepsilon_\alpha$ and $\varepsilon_1\varepsilon_2 = -\bar{\varepsilon}_1\bar{\varepsilon}_2 = 1$, then the relevant linear combinations are $\bar{Q}_1 + \zeta Q_2$ and $Q_1 + \zeta\bar{Q}_2$ with $\zeta \in \mathbb{C}^\times$. For definiteness we set

$$Q = \bar{Q}_1 + Q_2 \tag{3.87}$$

[3] Actually there is no fundamental reason that we must equate $U(1)_V$ and $U(1)_R$, as there can be a shift by a global $U(1)$ symmetry. However, if they are different, the action of Q near the poles can no longer be interpreted as the action of a supercharge of the twisted Ω-deformed theory.

and use this as a BRST operator. This squares to a rotation of the S^2 about the axis through the poles $\theta = 0$ and π, plus a vector R-rotation:

$$Q^2 = \frac{1}{r}\Big(\mathcal{L}_{\partial_\varphi} + \frac{1}{2}F_V\Big). \tag{3.88}$$

Near the north pole $\theta = 0$, the action of Q looks like that of a supercharge in the Ω-deformed, topologically twisted theory [24] on $\mathbb{R}^2_\varepsilon \times \mathbb{R} \times S^1$ with $\varepsilon = 1/r$. Near the south pole $\theta = \pi$, it looks like the action of the corresponding supercharge in the Ω-deformed theory with $\varepsilon = -1/r$, twisted in the opposite manner.

For such a theory we thus constructed, how would its effective theory correspond to a quantum integrable system? Let us explain as follows. Notably, since now the Q-invariant sector of the twisted theory is invariant under deformations of the Kähler structure of C, we can rescale the metric of C by a large factor. Then the theory at energies $\mu \ll 1/r$ is described by a two-dimensional theory on C which depends only on the conformal structure (for a given spin structure). The compactification of this two-dimensional conformal field theory on a circle can be identified with the quantum integrable system which we are after.

Let us clarify how this works. We can take C to be $\mathbb{R} \times S^1$, and consider the low-energy effective theory of the $\mathcal{N} = 2$ supersymmetric gauge theory we just constructed on $S^2 \times \mathbb{R} \times S^1$. Compactifying the 4d theory on the S^1, we can obtain a three-dimensional effective theory. Then, performing the dualization procedure we previously introduced, we should obtain a sigma model on $S^2 \times \mathbb{R}$. Finally, by localizing the the sigma model on S^2, we can obtain a quantum mechanical system which is the desired quantum integrable system.

However, instead of carrying out a straightforward dimensional reduction of the four-dimensional theory, in the following we shall use a different strategy. Since the three-dimensional theory possesses $\mathcal{N} = (2, 2)$ supersymmetry on the S^2, we can lift the sigma model on S^2 with $\mathcal{N} = (2, 2)$ supersymmetry to obtain the desired sigma model on $S^2 \times \mathbb{R}$, just as we did for the construction of the four-dimensional $\mathcal{N} = 2$ supersymmetric guage theory. Let us now move to the construction of such a sigma model.

3.2.2 Low-energy Effective Theory: The Sigma Model on $S^2 \times \mathbb{R}$

Here we are interested in the low-energy dynamics of the $\mathcal{N} = 2$ supersymetric gauge theory on $S^2 \times \mathbb{R} \times S^1$, which turns out to be described by a sigma model on $S^2 \times \mathbb{R}$, as explained below.

We are interested in the effective description of the theory on the Coulomb branch at energies $1/r \ll \mu \ll 1/R$ (where we take the radii, r of the S^2 and R of the S^1, to be sufficiently large). In such a case, as explained in the previous section, the dynamics can be described by a three-dimensional gauge theory on $S^2 \times \mathbb{R}$ which, roughly speaking, is the dimensional reduction of the four-dimensional theory on the S^1. As in the case of $\mathbb{R}^3$, we dualize the gauge fields in this theory to periodic scalars. This dualization is carried out following the procedure showed in the previous section (since $S^2 \times \mathbb{R}$ is topologically almost $\mathbb{R}^3$, only the origin removed), and produces an $\mathcal{N} = 4$ supersymmetric sigma model whose target space is the total space of a complex integrable system, denoted by $\mathcal{M} \to \mathcal{B}$.

This sigma model has $\mathcal{N} = (2,2)$ supersymmetry on S^2, since the $\mathcal{N} = (2,2)$ supersymmetry on the S^2 is unbroken under dimensional reduction as well as the dualization. Before dualization, the vector multiplet scalars a^I sit in gauge-invariant twisted chiral multiplet, commonly denoted as Σ [60]. After the dualization they are again part of twisted chiral multiplets, and moreover, the same is true for the holomorphic coordinates z_I of the fibers of $\mathcal{M}$. The reason is that, as we will see, in order to formulate the sigma model we need to turn on a (twisted) superpotential. The scalars a^I, z_I have vector R-charge $q = 0$, so any superpotential constructed out of them has $q = 0$. It follows that if they were part of untwisted chiral multiplets, then the superpotential would break $U(1)_V$ and hence supersymmetry. (A superpotential breaks $U(1)_V$ unless it has $q = 2$. By contrast, a twisted superpotential preserves $U(1)_V$ regardless of the vector R-charge.)

Thus, in summary, the low-energy dynamics of the theory on $S^2 \times \mathbb{R} \times S^1$ is described by an $\mathcal{N} = 4$ supersymmetric sigma model with hyperkähler target space $\mathcal{M}$, formulated on $S^2 \times \mathbb{R}$. It preserves $\mathcal{N} = (2,2)$ supersymmetry on the

S^2 and is constructed from twisted chiral multiplets. Our first task is to write down the action of this sigma model.

The sigma model

The strategy for constructing the action of the sigma model on $S^2 \times \mathbb{R}$ is basically the same as the one we employed for the four-dimensional supersymmetric gauge theory: first we write down the action for the two-dimensional theory on S^2, then we lift it to $S^2 \times \mathbb{R}$.

The two-dimensional theory is the dimensional reduction of the three-dimensional sigma model. It is an $\mathcal{N} = (2, 2)$ supersymmetric sigma model on S^2, with target space $\mathcal{M}$. Given holomorphic coordinates on $\mathcal{M}$, the map $\upsilon\colon S^2 \to \mathcal{M}$ of the sigma model can be described locally by complex scalars υ^i, $i = 1, \dots, 2r$. A choice of a local symplectic basis $\{\alpha_I, \beta^I\}$ of Γ provides holomorphic coordinates in the complex structure J_3, namely (a^I, z_I). As our purpose is to obtain the corresponding integrable system, let us focus on this complex structure.

The scalars υ^i are completed with Weyl spinors $\chi_+^{\bar{i}}$, χ_-^i, $\bar{\chi}_+^i$, $\bar{\chi}_-^{\bar{i}}$ and complex auxiliary fields E^i to form twisted chiral multiplets; the subscripts $\pm$ of the spinors indicate the chirality. Their supersymmetry transformations are [61, 62]

$$
\begin{aligned}
\delta \upsilon^i &= \bar{\varepsilon}_+ \chi_-^i + \varepsilon_- \bar{\chi}_+^i, \\
\delta \bar{\upsilon}^{\bar{i}} &= -\bar{\varepsilon}_- \chi_+^{\bar{i}} - \varepsilon_+ \bar{\chi}_-^{\bar{i}}, \\
\delta \chi_+^{\bar{i}} &= i \nabla\!\!\!\!/\,{}_+{}^- \bar{\upsilon}^{\bar{i}} \varepsilon_- - \bar{E}^{\bar{i}} \varepsilon_+, \\
\delta \chi_-^i &= i \nabla\!\!\!\!/\,{}_-{}^+ \upsilon^i \varepsilon_+ - E^i \varepsilon_-, \\
\delta \bar{\chi}_+^i &= -i \nabla\!\!\!\!/\,{}_+{}^- \upsilon^i \bar{\varepsilon}_- + E^i \bar{\varepsilon}_+, \\
\delta \bar{\chi}_-^{\bar{i}} &= -i \nabla\!\!\!\!/\,{}_-{}^+ \bar{\upsilon}^{\bar{i}} \bar{\varepsilon}_+ + \bar{E}^{\bar{i}} \bar{\varepsilon}_-, \\
\delta E^i &= i \bar{\varepsilon}_- \nabla\!\!\!\!/\,{}_+{}^- \chi_-^i + i \varepsilon_+ \nabla\!\!\!\!/\,{}_-{}^+ \bar{\chi}_+^i, \\
\delta \bar{E}^{\bar{i}} &= -i \bar{\varepsilon}_+ \nabla\!\!\!\!/\,{}_-{}^+ \chi_+^{\bar{i}} - i \varepsilon_- \nabla\!\!\!\!/\,{}_+{}^- \bar{\chi}_-^{\bar{i}}.
\end{aligned}
\tag{3.89}
$$

Here $\nabla\!\!\!\!/\,{}_+{}^-$, $\nabla\!\!\!\!/\,{}_-{}^+$ are the nonzero matrix elements of the Dirac operator $\nabla\!\!\!\!/$. Note that we are taking ε, $\bar{\varepsilon}$ to be commuting spinors.

The Lagrangian for the two-dimensional sigma model can be written compactly in terms of a Kähler potential K, which is a locally-defined function on $\mathcal{M}$ that gives the Kähler form $\omega_3 = ig_{i\bar{j}}\, \mathrm{d}\upsilon^i \wedge \mathrm{d}\bar{\upsilon}^{\bar{j}}$ by $\omega_3 = i\partial\bar{\partial}K$:

$$
\begin{aligned}
\mathcal{L}_{\tilde{C}} &= \frac{1}{2}\big(\bar{\mathcal{Q}}_1\bar{\mathcal{Q}}_2\mathcal{Q}_1\mathcal{Q}_2 + \mathcal{Q}_1\mathcal{Q}_2\bar{\mathcal{Q}}_1\bar{\mathcal{Q}}_2\big)K \\
&= \frac{1}{2}\bar{\mathcal{Q}}_1\bar{\mathcal{Q}}_2\big(g_{i\bar{j}}\chi_-^i\chi_+^{\bar{j}}\big) + \frac{1}{2}\mathcal{Q}_1\mathcal{Q}_2\big(g_{i\bar{j}}\bar{\chi}_+^i\bar{\chi}_-^{\bar{j}}\big).
\end{aligned}
\tag{3.90}
$$

Computing the supersymmetry variations and integrating out the auxiliary fields, we get

$$
\mathcal{L}_{\tilde{C}} = g_{i\bar{j}}\partial^\mu\upsilon^i\partial_\mu\bar{\upsilon}^{\bar{j}} - ig_{i\bar{j}}\not{D}_-{}^+\bar{\chi}_+^i\chi_+^{\bar{j}} - ig_{i\bar{j}}\chi_-^i\not{D}_+{}^-\bar{\chi}_-^{\bar{j}} + R_{i\bar{j}k\bar{l}}\chi_-^i\chi_+^{\bar{j}}\bar{\chi}_+^k\bar{\chi}_-^{\bar{l}}. \tag{3.91}
$$

The Dirac operator $\not{D}$ is coupled to the pullback of the metric connection of $\mathcal{M}$ by υ.

Thus now we have the two-dimensional Lagrangian in hand, and the next step is to perform the lifting. To lift the supersymmetry transformations to three dimensions, we simply allow the fields to vary along the extra x^3-direction; thus the form of the transformation rules remain unchanged from the formula 3.89.

To lift the action, in addition we need to integrate the two-dimensional action over the x^3-direction:

$$
S_{\tilde{C}} = \int_{\mathbb{R}} \mathrm{vol}_{\mathbb{R}} \int_{S^2} \mathrm{vol}_{S^2}\, \mathcal{L}_{\tilde{C}}. \tag{3.92}
$$

(The symbol vol denotes the volume form of the corresponding space.) However, for the three-dimensional Lagrangian, some terms are still missing from the action $S_{\tilde{C}}$ so obtained, such as kinetic terms involving derivatives along the x^3-direction. How to get these missing terms by lifting? As we shall see, these missing terms can be supplied by a twisted superpotential in the two-dimensional theory.

In our context, a twisted superpotential is a holomorphic functional $\tilde{W}$ on $\mathrm{Map}(\mathbb{R}, \mathcal{M})$, the space of maps from $\mathbb{R}$ to $\mathcal{M}$. The bosonic field $\upsilon\colon S^2 \times \mathbb{R} \to \mathcal{M}$ of the three-dimensional sigma model gives rise to a map $\tilde{\upsilon}\colon S^2 \to \mathrm{Map}(\mathbb{R}, \mathcal{B})$, and $\tilde{W}$ is to be understood as a functional of $\tilde{\upsilon}$. Then the twisted F-term is given by

$$\mathcal{L}_{\tilde{W}} = i\Big(E^i\partial_i\tilde{W} + \chi_-^i\bar{\chi}_+^j\partial_{ij}\tilde{W}) + \bar{E}^{\bar{i}}\partial_{\bar{i}}\tilde{W}^* + \bar{\chi}_-^{\bar{i}}\chi_+^{\bar{j}}\partial_{\bar{i}\bar{j}}\tilde{W}^*) + \frac{2}{r}\operatorname{Im}\tilde{W}\Big), \quad (3.93)$$

where $\partial_i\tilde{W} = \frac{\delta\tilde{W}}{\delta\upsilon^i}$. The superpotial term being added, the lifted three-dimensional action thus takes the form

$$S = S_{\tilde{C}} + \int_{S^2}\mathrm{vol}_{S^2}\,\mathcal{L}_{\tilde{W}}. \quad (3.94)$$

With the twisted superpotential $\tilde{W}$ turned on, integrating out the auxiliary fields produces the potential term $\|\delta\tilde{W}\|^2$. We want to choose $\tilde{W}$ in such a way that this potential provides the bosonic kinetic term involving x^3-derivatives.

We expect the holomorphic functional $\tilde{W}$ to be constructed from the holomorphic symplectic form Ω, since this is the only object associated with the khyperkähler structure of $\mathcal{M}$ that is holomorphic in J_3 and can be integrated in some manner to define a functional. The appropriate choice turns out to be the following. Under the variation $\tilde{\upsilon} \to \tilde{\upsilon} + \delta\tilde{\upsilon}$, we require $\tilde{W}$ to change by

$$\delta\tilde{W}(\tilde{\upsilon}) = \frac{i}{2}\int_{\mathbb{R}}\Omega_{ij}\delta\tilde{\upsilon}^i\mathrm{d}\tilde{\upsilon}^j. \quad (3.95)$$

With this choice, the potential

$$\|\delta\tilde{W}\|^2 = \frac{1}{4}\int_{\mathbb{R}}\mathrm{vol}_{\mathbb{R}}\,g^{i\bar{j}}\big(\Omega_{ik}\partial^3\upsilon^k\big)\big(\bar{\Omega}_{\bar{j}\bar{l}}\partial_3\bar{\upsilon}^{\bar{l}}\big). \quad (3.96)$$

Using the relations $\Omega = -i(\omega_1 + i\omega_2)$ and $\omega_\alpha = J_\alpha g$, we can rewrite this as

$$\frac{1}{4}\int_{\mathbb{R}}\mathrm{vol}_{\mathbb{R}}\,g\big((J_1+iJ_2)\partial^3\upsilon, \overline{(J_1+iJ_2)\partial_3\upsilon}\big) = \frac{1}{2}\int_{\mathbb{R}}\mathrm{vol}_{\mathbb{R}}\,g\big(\partial^3\upsilon, \partial_3\bar{\upsilon}\big). \quad (3.97)$$

In this equality we used the fact that the hermitian metric is compatible with the complex structure $(J_1 + iJ_2)/\sqrt{2}$. We see that this is precisely the missing bosonic kinetic term. So this is the right choice for $\tilde{W}$, up to an overall phase. It will become clear shortly that the phase is also right.

We now have to construct the holomorphic functional $\tilde{W}$ that satisfies the required property 3.95. We first assume that the cohomology class $[\Omega] = 0$ so that there exists a one-form λ such that $\Omega = \mathrm{d}\lambda$ (which is the case when the hypermultiplet masses are zero in the ultraviolet.). Then

$$\tilde{W}(\tilde{v}) = \frac{i}{2}\int_{\mathbb{R}} \tilde{v}^*\lambda \tag{3.98}$$

possesses the desired property.

Then, let us turn to the case that $[\Omega] \neq 0$, where the construction is a bit more involved and proceeds in the following three steps. First, we pick a representative $\tilde{v}_0([\tilde{v}])$ in each homotopy class $[\tilde{v}]$, which is a class of maps in $\mathrm{Map}(\mathbb{R}, \mathcal{M})$ that coincide with $\tilde{v}$ at $x^3 = \pm\infty$ and can be continuously deformed to $\tilde{v}$. Next, given $\tilde{v} \in \mathrm{Map}(\mathbb{R}, \mathcal{M})$, we choose a homotopy $\tilde{Y}\colon [0,1] \times \mathbb{R} \to \mathcal{M}$ between $\tilde{Y}_0 = \tilde{v}_0([\tilde{v}])$ and $\tilde{Y}_1 = \tilde{v}$. Finally, we set

$$\tilde{W} = \frac{i}{2}\int_{[0,1]\times\mathbb{R}} \tilde{Y}^*\Omega. \tag{3.99}$$

To verify that this definition indeed satisfies the condition (3.95), we can assume that $\delta\tilde{v}$ is supported in a sufficiently small neighborhood in $\mathcal{M}$ so that we can use a local expression $\Omega = \mathrm{d}\lambda$ to compute the variation. Then we get

$$\delta\tilde{W} = \frac{i}{2}\int_{[0,1]\times\mathbb{R}} \delta\big(\tilde{Y}^*\mathrm{d}\lambda\big) = \frac{i}{2}\int_{\mathbb{R}} \delta(\tilde{v}^*\lambda) = \frac{i}{2}\int_{\mathbb{R}} \Omega_{ij}\delta\tilde{v}^i \mathrm{d}\tilde{v}^j. \tag{3.100}$$

If we compactify the $\mathbb{R}$ to an S^1 and consider the contractible loops, $\tilde{W}$ reduces to the symplectic action functional $\tilde{W}_H$ for Hamiltonian $H = 0$, which plays a fundamental role in Floer homology.

However, the functional $\tilde{W}$ depends on the choice of the homotopy $\tilde{Y}$, so it is actually not single-valued. If we chose another homotopy $\tilde{Y}'$, then $\Delta\tilde{Y} = \tilde{Y}' - \tilde{Y}$ is a map from $S^1 \times \mathbb{R}$ to $\mathcal{M}$, and $\tilde{W}$ changes by

$$\Delta\tilde{W} = \frac{i}{2}\int_{S^1\times\mathbb{R}} \Delta\tilde{Y}^*\Omega. \tag{3.101}$$

Here, since in (3.93) $\mathcal{L}_{\tilde{W}}$ contains term $2i\,\mathrm{Im}\,\tilde{W}/r$, for the path integral to be well-defined the integral of $2i\,\mathrm{Im}\,\Delta\tilde{W}/r$ over the S^2 must be an integer multiple of $2\pi i$. The boundary conditions at infinity effectively collapse the two ends of the cylinder $S^1 \times \mathbb{R}$ to points, making a two-cycle. So by de Rham's theorem, this requirement is satisfied if

$$2r[\operatorname{Re}\Omega] \in H^2(\mathcal{M};\mathbb{Z}). \tag{3.102}$$

This can be viewed as the condition on the symplectic form in geometric quantization of the real symplectic manifold $(\mathcal{M}, \operatorname{Re}\Omega/\hbar)$, with $\hbar = 1/2r$. In our context, it means that the real part of the hypermultiplet masses must be quantized to integers in the unit of $\hbar$.

However, even though the problem of multi-valuedness is resolved, there are still two related ambiguities in the definition of $\tilde{W}$. One is associated with the choice of the representative paths $\tilde{v}_0$. The other is the values $\tilde{W}(\tilde{v}_0)$, which we can set freely since shifting them by constants does not affect the variation $\delta\tilde{W}$.

How to fix these ambiguities? Remember that the Lagrangian of the three-dimensional sigma model essentially comes from the three-dimensional effective gauge theory via the *dualization*. So by looking at how the term $2i\operatorname{Im}\tilde{W}/r$ arises via the dualization, we should be able to fix these ambiguities. The fixing is what we will do in the following, in the semiflat approximation.

Remember, in the dualization process, we added to the action of the effective gauge theory the term

$$-\frac{i}{2\pi}\int_{S^2\times\mathbb{R}} F^{(3),I}\wedge \mathrm{d}\theta_{m,I} = -\frac{i}{2\pi}\int_{S^2\times\mathbb{R}} \mathrm{vol}_{S^2\times\mathbb{R}}\, F^{(3),I}_{\hat{1}\hat{2}}\partial_{\hat{3}}\theta_{m,I} + \cdots . \tag{3.103}$$

where "$\cdots$" represent the terms involving the components of $F^{(3),I}$ other than $F^{(3),I}_{\hat{1}\hat{2}}$. On the other hand, on $S^2\times\mathbb{R}$ the Lagrangian (3.40) for flat spacetime should be modified. How to do the modification? By looking at the form of the Lagrangian (3.78) for the four-dimensional $\mathcal{N}=2$ theory on $S^2\times C$, (where $\mathcal{L}_V$ and $\mathcal{L}_\Phi$ are given by (3.76) and (3.77) respectively,) we deduce that the modified Lagrangian contains

$$\frac{R}{2}\operatorname{Im}\tau_{IJ}\left(F^{(3),I}_{\hat{1}\hat{2}}+\frac{\operatorname{Re}a^I}{r}\right)\left(F^{(3),J}_{\hat{1}\hat{2}}+\frac{\operatorname{Re}a^J}{r}\right)+\frac{i}{2\pi}\left(\operatorname{Re}\tau_{IJ}F^{(3),I}_{\hat{1}\hat{2}}+\operatorname{Im}\tau_{IJ}\frac{\operatorname{Im}a^I}{r}\right)\partial_{\hat{3}}\theta^J_e. \tag{3.104}$$

Integrating $F^I_{\hat{1}\hat{2}}$ out then produces the term

$$\frac{i}{2\pi r}\big[\operatorname{Re} a^I\big(\partial_{\hat{3}}\theta_{m,I} - \operatorname{Re}\tau_{IJ}\partial_{\hat{3}}\theta_e^J\big) + \operatorname{Im} a^I \operatorname{Im}\tau_{IJ}\partial_{\hat{3}}\theta_e^J\big]$$
$$= \frac{i}{2\pi r}\operatorname{Re}\big[a^I\big(\partial_{\hat{3}}\theta_{m,I} - \tau_{IJ}\partial_{\hat{3}}\theta_e^J\big)\big]. \quad (3.105)$$

We identify this with the desired $i\operatorname{Im}\tilde{W}/r$ (apart from a term involving $\partial_{\hat{3}}\tau_{IJ}$ which we have ignored in this analysis). Recalling the definition (3.59) of the holomorphic coordinates z_I, we see that $\tilde{W}$ can be written, locally on $\mathcal{M}$, as

$$\tilde{W} = \frac{i}{4\pi}\int_{\mathbb{R}} a^I \mathrm{d}z_I. \quad (3.106)$$

This formula satisfies the condition (3.95).[4] Thus the identification is indeed true, and the formula (3.106) fixes the aforementioned ambiguities. For the choice of representatives $\tilde{v}_0$, we can choose each of them to be a composition of "horizontal" paths along which $\mathrm{d}z_I = 0$, and "vertical" paths along the fibers above fixed points on $\mathcal{B}$. The value of $\tilde{W}(\tilde{v}_0)$ is equal to the sum of the values assigned to these component paths. For horizontal paths, $\tilde{W} = 0$, and for vertical paths, $\tilde{W}$ is given by a linear combination of a^I specified by the above formula.

3.3 Localization to the Quantum Integrable System

Thus far, we have written down the Lagrangian for the three-dimensional sigma model on $S^2\times\mathbb{R}$, so finally, we are ready to localize the path integral for this low-energy effective theory. Via the localization, the theory will reduce to a quantum mechanical system, corresponding the desired quantum integral system.

In order to perform the localization, the essentially necessary feature of the sigma model is that the relevant part (3.92) of the action is Q-exact. We indeed have such a feature: up to total derivatives we can write the twisted chiral multiplet Lagrangian (3.90) as:

$$\mathcal{L}_{\tilde{C}} = \frac{1}{2}\mathcal{Q}\big[\bar{\mathcal{Q}}_2\big(g_{i\bar{j}}\chi^i_-\chi^{\bar{j}}_+\big) - \mathcal{Q}_1\big(g_{i\bar{j}}\bar{\chi}^i_+\bar{\chi}^{\bar{j}}_-\big)\big], \quad (3.107)$$

[4] Recall that originally the formula (3.58) for Ω was obtained in the semiflat approximation, and then we went on to argue that there are no instanton corrections. We can now make the same statement more precisely as the nonrenormalization of $\tilde{W}$.

where we used the fact that $\{Q_\alpha, \bar{Q}_\alpha\}$ generates a rotation of the S^2, and the Kähler property of the target space metric.

By the Q-exactness of $S_{\tilde{C}}$, we can rescale it by an overall factor without affecting the physics of the theory, since all physical quantities belong to the Q-invariant sector. We rescale it as

$$\mathcal{L}_{\tilde{C}} \to t^2 \mathcal{L}_{\tilde{C}}, \tag{3.108}$$

and take the limit $t \to \infty$. Under such a rescaling, integrating out the auxiliary fields E leaves the superpotential terms in (3.93) which contain E to be proportional to $1/t$ thus vanish; and since (3.91) is rescaled by t, when $t \to \infty$, the path integration over υ receives contributions only from a neighbourhood of the configurations such that

$$\partial_\mu \upsilon^i = 0. \tag{3.109}$$

The path integral therefore localizes to the maps $\upsilon_0 \colon S^2 \times \mathbb{R} \to \mathcal{B}$ that are constant on the S^2.

By such a localization, we expand υ as $\upsilon = \upsilon_0 + \upsilon'$. Thus now the partition function can written as

$$Z = \int \mathcal{D}\upsilon_0 \int \mathcal{D}\upsilon' \mathcal{D}\chi \exp\bigl(-t^2 S_{\tilde{C}}(\upsilon_0, \upsilon') + \int_{S^2} \mathrm{vol}_{S^2} \mathcal{L}_{\tilde{W}}(\upsilon_0, \upsilon'; E = 0)\bigr), \tag{3.110}$$

where $S_{\tilde{C}}$ and $\mathcal{L}_{\tilde{W}}$ are given by (3.92) and (3.93) respectively. Note that since on S^2 there exist *no* covariant constant spinors, there are *no* fermion zero modes.

To evaluate the path integral, we rescale υ' and the fermions by a factor of $1/t$. After doing this, in the limit $t \to \infty$, it is clear that only the quadratic terms of $\mathcal{L}_{\tilde{C}}$ survive, while the higher order terms accompanied with an overall factor proportional to the power of $1/t$ vanish. Thus the left terms are

$$\mathcal{L} = g_{i\bar{j}}(\upsilon_0)\partial^\mu \upsilon'^i \partial_\mu \bar{\upsilon}'^{\bar{j}} - i g_{i\bar{j}}(\upsilon_0) \not{D}_-{}^+ \bar{\chi}^i_+ \chi^{\bar{j}}_+ - i g_{i\bar{j}}(\upsilon_0) \chi^i_- \not{D}_+{}^- \bar{\chi}^{\bar{j}}_-, \tag{3.111}$$

which depends on the background υ_0. However, for each background υ_0 and at each point on the $\mathbb{R}$, we can find Kähler normal coordinates such that $g_{i\bar{j}}(\upsilon_0) =$

δ_{ij} and $\delta_k g_{i\bar{j}}(v_0) = \partial_{\bar{k}} g_{i\bar{j}}(v_0) = 0$. Thus the left terms in the Lagrangian become

$$\sum_i \left(\partial^\mu v'^i \partial_\mu \bar{v}'^{\bar{i}} - i\partial\!\!\!/_{-}{}^{+}\bar{\chi}^i_{+}\chi^{\bar{i}}_{+} - i\chi^i_{-}\partial\!\!\!/_{+}{}^{-}\bar{\chi}^{\bar{i}}_{-}\right), \tag{3.112}$$

independent of v_0. Therefore, the path integral over v' and the fermions just produces a constant, which we absorb in the measure.

The final step in the path integral is to integrate over all possible backgrounds v_0. Since they are constant on the S^2, the integration of v_0 over the S^2 just gives a factor of $4\pi r^2$. Then, from (3.110) we in the end obtain the following path integral:

$$\int \mathcal{D}v_0 \exp\Bigl(\frac{i}{\hbar}S(v_0)\Bigr), \tag{3.113}$$

where

$$S(v_0) = -4\pi \operatorname{Im}\tilde{W}, \quad \hbar = \frac{1}{2r}, \tag{3.114}$$

in which $S(v_0)$ comes from the last term in (3.93). Locally on $\mathcal{B}$, the action is expressed as

$$S = -\int_{\mathbb{R}} \operatorname{Re}\bigl(a^I \mathrm{d}z_I\bigr) = -\int_{\mathbb{R}} \bigl(\operatorname{Re} a^I \mathrm{d}\theta_{m,I} - \operatorname{Re} a_{D,I}\, \mathrm{d}\theta^I_e\bigr), \tag{3.115}$$

where we used the boundary conditions $\mathrm{d}a^I = 0$ at infinity to obtain the last expression.

Obviously, what we obtained in (3.113) is the path integral of a quantum mechanical system, and it quantizes the following classical integrable system. The action (3.115) is the one for the real integrable system $(\mathcal{M}, \operatorname{Re}\Omega)$, written in action-angle variables; there are $2r$ commuting action variables $\operatorname{Re} a^I$, $\operatorname{Re} a_{D,I}$, and $2r$ commuting angle variables $\theta_{m,I}$, θ^I_e. As we have shown, the path integral of the Q-invariant sector of the effective theory reduces to the path integral (3.113) quantizing this classical integrable system. Therefore, we successfully obtain the corresponding quantum integrable system describing the low-energy dynamics of the Q-invariant sector.

Lastly, we would like to show that semiclassically that the quantum integrable system we obtained reproduces the vacuum structure of the $\mathcal{N} = 2$ gauge theory on $S^2 \times \mathbb{R} \times S^1$. Suppose that we fix the holonomies θ^I_e (i.e. let $\mathrm{d}\theta^I_e$) at infinity. Remind that on the curved spacetime $S^2 \times \mathbb{R} \times S^1$, the gauge kinetic term $\mathrm{Tr}\, F_{\hat{1}\hat{2}}^2$ is shifted to $\mathrm{Tr}(F_{\hat{1}\hat{2}} + \mathrm{Re}\,\phi/r)^2$ in the ultraviolet Lagrangian (3.84). Therefore, the flux quantization condition (3.41) shall be shifted, and as a result, the effect of the curvature to the vacuum moduli is that a^I must satisfy

$$\mathrm{Re}\, a^I \in \frac{\mathbb{Z}}{2r}. \tag{3.116}$$

On the other hand, in the quantum integrable system, by integrating over the periodic scalars $\theta_{m,I}$, we obtain the constraint

$$\frac{\mathrm{Re}\, a^I}{\hbar} \in \mathbb{Z}, \tag{3.117}$$

which exactly recovers (3.116). If we instead chose to fix $\theta_{m,I}$ and integrate over θ_e^I, then we would get the electromagnetic dual of the above constraint. Thus we conclude that semiclassically the quantum integrable system indeed reproduces the vacuum structure of the $\mathcal{N} = 2$ gauge theory on $S^2 \times \mathbb{R} \times S^1$.

3.4 The Hemisphere Case: Nekrasov and Shatashvili Correspondence

Let us recall what we have obtained thus far. We constructed the low energy effective theory of $\mathcal{N} = 2$ supersymmetric gauge theory on $S^2 \times \mathbb{R} \times S^1$. This effective theory is a sigma model on $S^2 \times S^1$, with the $\mathcal{N} = (2, 2)$ supersymmetry on the S^2. This sigma model can be obtained via dualization of the three-dimensional low-energy effective gauge theory on the $S^2 \times \mathbb{R}$ (which is obtained by the dimensional reduction of the four-dimensional low-energy effective gauge theory on the S^1). By localization on the S^2, the sigma model reduces to the path integral of a quantum mechanical system, which exactly corresponds to a quantum integrable system. Such an integrable system is quantized by the Planck parameter $\hbar$ in the path integral, where

$$\hbar = \frac{1}{2r}, \tag{3.118}$$

with r the radius of the S^2.

In this section, we would like to extend our finding to relating to some well-known results [23, 25]. To this end, we replace the sphere S^2 with a hemisphere D^2 of radius r, and using the same localization method, we find that this new setup gives us a variant of the correspondence discovered by Nekrasov and Shatashvili [23]. Let us reveal this result in the following.

To perform the replacement, firstly recall that the square of our supercharge $Q = \bar{Q}_1 + Q_2$ generates a rotation of the S^2, so we take D^2 to be invariant under this rotation. Given this, it is clear that the supersymmetry transformations and the supersymmetric Lagrangian are the same as in the S^2 case. The new feature is that the spacetime of D^2 has a boundary, so we have to specify a boundary condition that preserves Q. We also demand that the boundary condition preserves the rotational symmetry of D^2. Since $\bar{Q}_1$ and Q_2 have opposite charges under the rotation, such boundary conditions preserve these supercharges separately. Thus they are half-BPS boundary conditions of the $\mathcal{N} = (2, 2)$ supersymmetry, describing half-BPS branes in the target space. $\mathcal{N} = (2, 2)$ supersymmetric gauge theories on a hemisphere with half-BPS boundary conditions have recently been studied in [63, 64, 71].

For our interest, we consider the branes described as follows. Recall that by (3.58), the Kähler two form $\omega_1 = -\operatorname{Im}\Omega$, where the symplectic form $\Omega \propto \mathrm{d}a^I \wedge \mathrm{d}\theta_{m,I} - \mathrm{d}a_{D,I} \wedge \mathrm{d}\theta_e^I$. Therefore, with respect to ω_1, we can define two Lagrangian submanifolds $\mathcal{L}_1$, $\mathcal{L}_2 \subset \mathcal{M}$ by

$$\mathcal{L}_1\colon\ \operatorname{Im} a_{D,I} = 0 = \theta_{m,I}, \tag{3.119}$$

$$\mathcal{L}_2\colon\ \operatorname{Im} a^I = 0 = \theta_e^I. \tag{3.120}$$

And we are interested in the branes supported on these two Lagrangian submanifolds. In the semiflat approximation one can check that $\mathcal{L}_1$ and $\mathcal{L}_2$ are holomorphic under the complex structure J_2 and Lagrangian with respect to ω_3. By studying the same kinds of branes, Nekrasov and Witten [25] established a connection between $\mathcal{N} = 2$ supersymmetric gauge theories on the Ω-deformed spacetime $\mathbb{R}^2_\varepsilon \times \mathbb{R} \times S^1$ and quantum integrable systems. We shall see that a similar connection can be established in the present setup.

Via localization, just as in the S^2 case, we can show that the Q-invariant sector of the low-energy effective theory on $D^2 \times \mathbb{R} \times S^1$ reduces to a quantum integrable system. (The difference is that in the D^2 case the field configurations are constrained by the boundary condition we chose.) The path integral localizes to the configurations v_0 that are constant on D^2 and therefore determined by the boundary value. These are maps from $\mathbb{R}$ to $\mathcal{L} \subset \mathcal{B}$, where $\mathcal{L} = \mathcal{L}_1$ or $\mathcal{L}_2$ depending on the choice of the boundary condition. The one-loop determinants are still independent of the background configuration v_0 and can be absorbed in the measure. Hence, the localization leads to the same expression (3.113). Note that we get $S(v_0)$ by integrating out the volume of the D^2, and then the value of $S(v_0)$ is half of that in the S^2 case, as the area of the D^2 is half. Therefore, compared to the S^2 case, the differences in the expression (3.113) are that the integration domain is now $\mathrm{Map}(\mathbb{R}, \mathcal{L})$ and the Planck constant is twice the previous value:

$$\hbar = \frac{1}{r}. \tag{3.121}$$

Thus, we conclude that the result of the localization is the path integral for a quantum integrable system that quantizes the real integrable system $(\mathcal{L}, \operatorname{Re}\Omega)$.

Moreover, a further relation can be established between our setup and Nekrasov and Witten's setup as follows. First note that the Hilbert space of the quantum integrable system is associated to a "time slice" at fixed x^3, so the physical states are described in the gauge theory as Q-invariant functionals of the field configurations over $D^2 \times \{x^3\} \times S^1$. We would like to recast these states to states of open strings stretched between two branes. In order to do this recasting, we first reduce the theory on the S^1, and by virtue of the $U(1)$ rotational symmetry of the D^2, we further reduce the theory on the circle fibers of the D^2. The second reduction turns D^2 into an interval $I = [0, r]$. Therefore by the reductions the theory becomes a sigma model on $I \times \mathbb{R}$. And we now have two branes located at the two ends of I. One of them is the brane we placed on the boundary of D^2, given by the boundary condition; the other, new brane sits at the end that was formerly the pole of D^2. The latter is a space-filling brane since at the pole the field configurations were not constrained in any submanifold of $\mathcal{M}$. Therefore, via the reductions, the gauge theory states are turned into open string states stretched between these two branes. We then see here a close parallel to the construction of Nekrasov and Witten; in their construction, by the reduction of the Ω-deformed theory on the circle fibers of a cigar-shaped manifold (which looks much like a hemisphere near the tip), one can arrive at a topological sigma model on $\mathbb{R} \times I$ with target space $\mathcal{M}$, and the Hilbert space of the quantum integrable system is obtained as the space of open strings stretched between a space-filling (A, B, A)-brane and a middle-dimensional (A, B, A)-brane located at the ends of I.

To further illustrate the property of the quantum integrable system, we take $\mathcal{L} = \mathcal{L}_1$ as an example. In this case, by (3.115), the action of the quantum integrable system is then

$$S = \int_{\mathbb{R}} \operatorname{Re} a_{D,I} \, \mathrm{d}\theta_e^I. \tag{3.122}$$

Since the $\operatorname{Re} a_{D,I}$ commute with one another, states are labeled by their eigenvalues. Integrating over the periodic scalars θ_e^I imposes the constraint

$$\frac{\operatorname{Re} a_{D,I}}{\hbar} \in \mathbb{Z} \tag{3.123}$$

on the possible values of these parameters. In view of the fact that $\operatorname{Im} a_{D,I} = 0$ on $\mathcal{L}$, this condition can be written as

$$r a_{D,I} = r \frac{\partial \mathcal{F}(a; r)}{\partial a^I} \in \mathbb{Z}, \tag{3.124}$$

This is the Bethe ansatz equation with Yang-Yang function $Y = r\mathcal{F}/2\pi i$. Thus the spectrum of the quantum integrable system is determined by the effective prepotential in the form of the Bethe ansatz equation.

What we have just found is a variant of the correspondence discovered by Nekrasov and Shatashvili [23]. The Ω-deformed spacetime $\mathbb{R}^2_\varepsilon \times \mathbb{R} \times S^1$ reduces in the infrared to a two-dimensional gauge theory on $\mathbb{R} \times S^1$. If we write $\mathcal{W}(a; \varepsilon)$ for the twisted superpotential of this theory, then the equation that determines the vacua is

$$\frac{\partial \mathcal{W}(a; \varepsilon)}{\partial a^I} \in i\mathbb{Z}. \tag{3.125}$$

The Nekrasov-Shatashvili correspondence identifies $\mathcal{W}$ with the Yang-Yang function of the quantum integrable system.[5] We see that $\mathcal{W}$ plays the role of $r\mathcal{F}$ in our correspondence.

The two correspondences agree in the limit $r \to \infty$ and $\varepsilon \to 0$. In the limit $\varepsilon \to 0$, the twisted superpotential behaves as

$$\mathcal{W}(a;\varepsilon) = \frac{i\mathcal{F}(a;\varepsilon = 0)}{\varepsilon} + \cdots , \tag{3.126}$$

where $\mathcal{F}(a;\varepsilon)$ is the effective prepotential of the Ω-deformed theory, and $\cdots$ denotes terms regular in ε. Since $\mathcal{F}(a;\varepsilon = 0)$ is the effective prepotential on flat spacetime $\mathbb{R}^3 \times S^1$ and therefore equals $\mathcal{F}(a;r = \infty)$, their correspondence coincides with ours in this limit under the identification $\varepsilon = 1/r$.

[5] In their case the correspondence can be established by considering a topological field theory, so the states of the quantum integrable system have zero energy and correspond to the vacua of the gauge theory. This is not the case for us, even though the action (3.122) appears to suggest that the Hamitonian is zero. The reason is that in the localization of path integral we ignored the ratio of the one-loop determinants, which shifts the Lagrangian by a zero-point energy. The energy becomes zero only in the limit $r \to \infty$, where the determinants for scalars and spinors are equal.

Chapter 4

Deciphering 3d/3d Correspondence via 5d SYM

4.1 Introduction

As introduced in chapter 1, since the BPS sector which preserves the topologically twisted supercharge is protected against dimensional reductions, two different theories in lower dimensions that are reduced from a topologically twisted theory in higher dimensions are equivalent to each other under identification of Q-invariant quantities. Thus we are able to reveal various correspondences in physics using topologically twisted theories. This is the direction that the present chapter takes. In this chapter, we will apply a (partially) topologically twisted five-dimensional super-Yang-Mills to gain a deeper understanding of a correspondence between two three-dimensional theories.

4.1.1 Background and Motivation

In the past several years, an intriguing correspondence between two classes of quantum field theories has been found, one being 3d $N = 2$ SCFTs and the other being 3d Chern-Simons theories with complex gauge group [31–34]. The 3d/3d correspondence belongs to the set of various correspondences between supersymmetric theories in d dimensions and nonsupersymmetric theories in $6 - d$ dimensions, to which the celebrated 2d/4d AGT correspondence also belongs [26]. In a similar manner to how the AGT correspondence was developed, after the establishment of the 3d/3d correspondence by directly comparing the partition functions of the two classes of theories, a significant amount of effort was subsequently put into understanding the correspondence from a higher dimensional viewpoint.

In this viewpoint, the $d/(6-d)$ correspondences are understood as follows. One starts with the $N = (2,0)$ SCFT in 6 dimensions formulated on the product $X \times M$ of a d-dimensional space X and a $(6-d)$-dimensional space M. One considers M to be a fairly general manifold and topologically twists the 6d theory along M, while choosing X to be equipped with specific geometry such that certain supercharges can be preserved without topologically twisting them along X. Given this, on one side, compactification of the theory on M produces a supersymmetric theory $T[M]$ on X; while on the other side, by localization on X, the theory reduces to a nonsupersymmetric theory $T[X]$ on M. Identifying protected quantities that are invariant under both procedures, one expects to establish a correspondence between $T[M]$ and $T[X]$. In the $d = 3$ case that we are interested in, $T[M]$ is a 3d $N = 2$ SCFT and $T[X]$ is a 3d complex Chern-Simons theory.

To date, for $d = 3$, despite the complexity of explicitly carrying out the compactification, deriving $T[X]$ on the localization side has been more or less achieved by various works [40–43]. This chapter is also dedicated to trying to decipher the 3d/3d correspondence from the 6d viewpoint, using a typical yet fresh setup. Compared to the previous works, the novelty of our construction is that we equip X with an Ω-background. We place the theory on $(S^1 \times_\varepsilon \mathbb{D}^2) \times M$, where $\mathbb{D}^2$ denotes a disk and ε is the Ω-deformation parameter. Using this setup, by localization on X we can obtain $T[X]$ as the *holomorphic* part of a complex Chern-Simons theory. We will see that in our setup the Ω-background is crucial for obtaining this result.

In this setup, however, as one can point out, there is one fatal obstacle to solve: localization computations require a Lagrangian description, but for the 6d $N = (2,0)$ theory no Lagrangian is known, so where can we actually start from? In our favour, the problem can be avoided since X contains an S^1. Compactifying the $(2,0)$ theory on the S^1 down to 5d, it gives us 5d super-Yang-Mills theory (SYM) [1] on $\mathbb{D}^2_\varepsilon \times M$. This 5d SYM is topologically twisted along M, while on

[1]According to Lambert et. al.'s conjecture [65–67], this is possibly true. That is, they conjectured that the 6d (2, 0) theory on S^1 is equivalent to the 5d super-Yang-Mills theory

the $\mathbb{D}^2$ the preserved supercharge can be viewed as the Ω-deformation of the B-twisted $N = (2,2)$ supercharges, as will be shown in section 3.

This 5d SYM does have a Lagrangian; starting with it, given suitable boundary conditions on the $\mathbb{D}^2$, the localization of the theory on $\mathbb{D}^2$ can produce the desired complex Chern-Simons theory. The procedure is shown in the following diagram:

$$\begin{tikzcd}
& \text{(2, 0) theory on } S^1 \times_\varepsilon \mathbb{D}^2 \times M \arrow[d, "\text{compactifying on } S^1"] & \\
& \text{SYM on } \mathbb{D}^2_\varepsilon \times M \arrow[dl, "\text{compactifying on } M"'] \arrow[dr, "\text{localizing on } \mathbb{D}^2_\varepsilon"] & \\
T(M) \text{ on } \mathbb{D}^2_\varepsilon & & T(X) \text{ on } M
\end{tikzcd}$$

To this end, primarily, we need to construct such a 5d SYM on $\mathbb{D}^2_\varepsilon \times M$. Our construction follows the "lifting operation" whose meaning will be clear in section 3. The transformation laws and Lagrangian of the 5d theory are obtained by lifting their counterparts in a 2d theory. In our context, the 2d theory is the Ω-deformation of the B-twisted $N = (2,2)$ gauge theory on $\mathbb{D}^2$. Our work is the first to construct such a 2d Ω-deformed B-twisted gauge theory, the explicit formulation of which was lacking thus far. Besides being used as a tool to construct the 5d theory, as we shall see, it is interesting in itself and should have other vast applications.

Starting with the 5d SYM thus constructed, we successfully derive $T[X]$ as the holomorphic part of a complex Chern-Simons theory; on the other side, unfortunately, deriving $T[M]$ [2]via compactification on M is much more difficult,

with *no additional UV degrees of freedom*. But this conjecture is yet to be proved or disproved. So here we actually make an assumption: in our case, the 6d theory can reduce to the 5d SYM with all physical information captured, *in the Q-invariant sector*. At the end of the chapter, we will see that our result is consistent with this assumption and in turn, it suggests that this assumption is indeed true.

[2]Here the $T[M]$ on $\mathbb{D}^2_\varepsilon$ can be considered as the compactification on S^1 of the corresponding 3d SCFT on $S^1 \times_\varepsilon \mathbb{D}^2$. This kind of $T[M]$ was studied by Dimofte et al. in [68], where, in a different approach, they established a correspondence between the vortex partition function of their 2d theory and the partition function of the complex Chern-Simons theory.

but hopefully we can carry it out in the future so that we can obtain $T(M)$ explicitly in our setup.

Despite such incompleteness, in our setup, some general properties of the $T[M]$ can be depicted and consequently, our result indicates that a mirror symmetry in two-dimensional Ω-deformed gauge theories should exist.

4.1.2 Outline

To summarize the chapter, in section 2, we construct the Ω-deformation of the B-twisted $N = (2,2)$ gauge theory on $\mathbb{D}^2$. It will be used as a tool to construct the 5d theory in section 3. We discuss the theory in detail and explore a fraction of its possible applications in the last part of section 2.

In section 3, we construct the 5d SYM on $\mathbb{D}^2_\varepsilon \times M$, by "lifting" the 2d theory constructed in section 2. Then given suitable boundary conditions, by localization on $\mathbb{D}^2$, we reduce the partition function of the 5d theory to

$$Z_{\text{5d}} \sim \int D\mathcal{A}\text{exp}\left(-\frac{\text{i}\pi}{\varepsilon}\int_M \text{Tr}(\mathcal{A}\wedge \text{d}\mathcal{A} - \frac{2\text{i}}{3}\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A})\right), \tag{4.1}$$

where $\mathcal{A}$ are complex gauge fields and ε is the Ω-deformation parameter. We thus derive the holomorphic part of the 3d complex Chern-Simons theory.

In section 4, we interpret the results we obtained about the Ω-deformed twisted 5d MSYM theory from the point of view of the 3d-3d correspondence. Following this we establish the correspondence between the 3d $\mathcal{N} = 2$ superconformal theory $T[M]$ and analytically continued Chern–Simons theory on M. Furthermore, our construction of the 5d theory, together with the 3d-3d correspondence, leads to a mirror symmetry between Ω-deformed 2d theories.

4.2 The Ω-deformation of 2d B-twisted Gauge Theory

In this section we construct the Ω-deformation of the B-twisted gauge theory described in the introduction. Before writing down its supersymmetry transformations and Lagrangian, let us first spell out the strategy that helps us to achieve this goal.

As in the case considered by Nekrasov and Okounkov in [69], a general Ω-deformed supersymmetric gauge theory, in brief, can be understood in the following way. A topologically twisted $N = 2$ supersymmetric gauge theory has a single scalar supercharge Q, which satisfies $Q^2 = 0$ and is used as a BRST operator. By contrast, the BRST operator Q of an Ω-deformed theory satisfies

$$Q^2 = L_V, \tag{4.2}$$

where L_V is the conserved charge that acts on fields as the Lie derivative $\mathcal{L}_V$, and V is a vector field that one chooses to generate an isometry of the spacetime manifold with respect to a given metric.

In our context, we can obtain such a BRST operator as follows. Let us recall that the full $N = (2,2)$ supersymmetry on the worldsheet $\Sigma = \mathbb{C}$, in the B-twisted form, consists of two scalar supercharges $\bar{Q}_+, \bar{Q}_-$ and a one-form supercharge $G = G_z \mathrm{d}z + G_{\bar{z}} \mathrm{d}\bar{z}$. The commutators of these charges give us

$$\{\bar{Q}_-, G_z\} = P_z,\ \{\bar{Q}_+, G_{\bar{z}}\} = P_{\bar{z}}, \tag{4.3}$$

where $P = P_z \mathrm{d}z + P_{\bar{z}} \mathrm{d}\bar{z}$ is the generator of translations. Therefore, if we choose a vector field $V = V^z \partial_z + V^{\bar{z}} \partial_{\bar{z}}$ with constant components V^z and $V^{\bar{z}}$ and define

$$Q = \bar{Q}_+ + \bar{Q}_- + \iota_V G, \tag{4.4}$$

we get the desired BRST operator Q satisfying $Q^2 = \iota_V P$.

Guided by this strategy, we go on to elucidate how we construct the Ω-deformed B-twisted gauge theory explicitly.

Our construction starts with the $N = (2,2)$ supersymmetry transformation laws for the B-twisted vector and chiral multiplets. Here we consider the Euclidean worldsheet Σ, whose symmetry group is the the rotation group $SO(2)_E = U(1)_E$. The $N = (2,2)$ supersymmetry also consists of two R-symmetries, the vector R-symmetry $U(1)_V$ and the axial R-symmetry $U(1)_A$. The B-twisting is done by replacing the rotation group $U(1)_E$ by the diagonal subgroup $U(1)'_E$ of $U(1)_E \times U(1)_A$.

After the B-twisting, 2 supercharges $\bar{Q}_\pm$ become scalars, while their conjugates $Q_\pm$ become the components of the one-form supercharge $G = Q_- \mathrm{d}z + Q_+ \mathrm{d}\bar{z}$. The B-twisted vector multiplet consists of one-form fields A and σ, one-form fermion λ, scalar fermions χ and $\bar{\chi}$, and scalar auxiliary field $\mathtt{D}$; the B-twisted chiral multiplet consists of complex scalar bosons ϕ and $\bar{\phi}$, one-form fermion ρ, scalar fermions η and $\bar{\eta}$, and auxiliary fields $\mathtt{F}$ and $\bar{\mathtt{F}}$.

In this context, provided Σ is flat, we first attempt to define a supercharge

$$Q = \bar{Q}_+ + \bar{Q}_- + V^z Q_- + V^{\bar{z}} Q_+. \tag{4.5}$$

Then, acted by Q, we can write down the supersymmetry transformation laws for the B-twisted component fields, using the standard $N = (2, 2)$ supersymmetry transformation laws [70]. Finally, *after some redefinitions of fields*, we obtain our desired supersymmetry transformation laws, which are independent of the metric and thus valid for an arbitrary Riemann worldsheet Σ, as will be shown in the following section.

4.2.1 Supersymmetry transformations and action

Supersymmetry transformation

Let us write down the supersymmetry transformation laws.

For the vector multiplet, the supersymmetry transformation laws read

$$\begin{aligned}
\delta A &= \mathrm{i}\lambda, \\
\delta\sigma &= \lambda + \iota_V \zeta, \\
\delta\lambda &= -\mathrm{i}\iota_V F_A + \mathrm{d}_A \iota_V \sigma, \\
\delta\zeta &= \mathrm{i}F_A + \mathrm{d}_A\sigma - \sigma\wedge\sigma, \\
\delta\alpha &= \mathrm{d}_A\sigma + \mathrm{i}\mathtt{D}, \\
\delta\mathtt{D} &= -\mathrm{i}\mathrm{d}_A(\iota_V\alpha) + \mathrm{i}[\iota_V\sigma, \alpha] + \mathrm{i}\mathrm{d}_A\lambda - \mathrm{i}[\sigma, \lambda] + \mathrm{i}\mathrm{d}_A(\iota_V\zeta).
\end{aligned} \tag{4.6}$$

Here, the bosonic fields A and σ are both one-forms; meanwhile for the fermions, λ is a one-form, while ζ and α, like the auxiliary field $\mathtt{D}$, are two-forms. Note that in our construction ζ and α can be respectively viewed as the Hodge duals of the antisymmetric and symmetric combinations of the two standard B-twisted scalar fermions. V is the vector field generating an isometry of the worldsheet Σ, while ι_V and $\star$ denote the interior product and Hodge star operator respectively.

With the Killing vector condition $\nabla_{\bar{z}}V^z = \nabla_z V^{\bar{z}} = 0$, one can verify that the following relation is true:

$$\delta^2 = \mathcal{L}_V - G(\iota_V\sigma), \tag{4.7}$$

where $\mathcal{L}_V$ is the gauge covariant Lie derivative by V. It acts as

$$\mathcal{L}_V = \mathrm{d}_A\iota_V + \iota_V\mathrm{d}_A \tag{4.8}$$

where

$$\mathrm{d}_A = \mathrm{d} - \mathrm{i}A, \tag{4.9}$$

on $\mathfrak{g}$-valued forms. Here $G(X)$ is the gauge transformation by $X \in \mathfrak{g}$ which acts on the fields as

$$G(X)A = -\mathrm{i}\mathrm{d}_A X, \quad G(X)\Phi = [X, \Phi], \tag{4.10}$$

where A denotes the gauge field, while Φ denote the other.

For the chiral multiplet, the transformation laws read

$$\begin{aligned}
\delta\phi &= \iota_V\rho, \\
\delta\bar{\phi} &= \bar{\eta}, \\
\delta\rho &= \mathrm{d}_A\phi - \sigma\phi + \iota_V\mathtt{F}, \\
\delta\bar{\eta} &= \iota_V\mathrm{d}_A\bar{\phi} + \bar{\phi}\iota_V\sigma, \\
\delta\bar{\mu} &= \bar{\mathtt{F}}, \\
\delta\mathtt{F} &= \mathrm{d}_A\rho - \sigma\wedge\rho + \zeta\phi, \\
\delta\bar{\mathtt{F}} &= \mathrm{d}_A\iota_V\bar{\mu} + \bar{\mu}\iota_V\sigma.
\end{aligned} \tag{4.11}$$

Here the bosons ϕ and $\bar{\phi}$ are complex scalar fields. As for fermions, ρ is a one-form, while $\bar{\eta}$ is a scalar and $\bar{\mu}$ is a two-form. The auxiliary fields $\mathtt{F}$ and $\bar{\mathtt{F}}$ are two forms. Here $\bar{\mu}$ and $\mathtt{F}$ can be viewed as the Hodge duals of the antisymmetric combinations of the standard B-twisted scalar fermions and auxiliary fields, while $\bar{\eta}$ and $\mathtt{F}$ can be viewed as the symmetric combinations, respectively.

One can check that the relation (4.7) also holds for the transformation laws for the chiral multiplet.

Action

With the transformation laws written down, let us move forward to constructing our Lagrangian.

The Q-invariant action consists of three parts:

$$S = S_{\mathrm{V}} + S_{\mathrm{C}} + S_{\mathrm{W}} \tag{4.12}$$

The first two parts S_{V} and S_{C} are both Q-exact, respectively representing the actions for vector and chiral multiplets, and the last part S_{W} is non-Q-exact, denoting the superpotential term for the chiral multiplet.

For the vector multiplet, we define the action by

$$\begin{aligned} S_{\mathrm{V}} =& \int_\Sigma \delta \,\mathrm{Tr}\big(\alpha \wedge \star(-\mathrm{d}_A \sigma + \mathrm{i}\mathtt{D} + 4\bar{D}\sigma) + \zeta \wedge \star\overline{\delta\zeta}\big) \\ =& \int_\Sigma \mathrm{d}^2 z\sqrt{h}\ \mathrm{Tr}\,\big(- F_{z\bar{z}}F^{\bar{z}z} + 2D_z\sigma_{\bar{z}}D^z\sigma^{\bar{z}} + 2D_z\sigma_z D^z\sigma^z + \mathcal{R}\sigma_z\sigma^z + [\sigma_z, \sigma_{\bar{z}}][\sigma^{\bar{z}}, \sigma^z] + \mathtt{D}'_{z\bar{z}}\mathtt{D}'^{\bar{z}z} \\ & - 2\nabla_z(\sigma_z D^z\sigma^z) + 2\nabla_{\bar{z}}(\sigma_z D^{\bar{z}}\sigma^z)\big) + S_{\mathrm{V}f}, \end{aligned} \tag{4.13}$$

where $\mathtt{D}'_{z\bar{z}} = -\mathtt{D}_{z\bar{z}} - 2\mathrm{i}D_{\bar{z}}\sigma_z$ and $\mathcal{R}$ is the Ricci scalar. Note $\mathcal{R} = 0$ if we chose the worldsheet to be a flat disk. Note that $F_{z\bar{z}}$ is antihermitian, and

$$2\,\mathrm{Tr}(D_z\sigma_{\bar{z}}D^z\sigma^{\bar{z}} + D_{\bar{z}}\sigma_{\bar{z}}D^{\bar{z}}\sigma^{\bar{z}}) = \mathrm{Tr}\,D_\mu\sigma_\nu D^\mu\sigma^\nu. \tag{4.14}$$

Here the fermionic part in the action reads

$$
\begin{aligned}
S_{\mathrm{V}f} &= -\int_\Sigma \mathrm{Tr}\left(\alpha\wedge\star\delta(-\mathrm{d}_A\sigma + \mathrm{i}\mathtt{D} + 4\bar{D}\sigma) + \zeta\wedge\star\delta\overline{\delta\zeta}\right) \\
&= -\int_\Sigma \mathrm{d}^2z\sqrt{h}\,\mathrm{Tr}(2\alpha^{\bar{z}z}D_z\lambda_{\bar{z}} + 2\alpha^{\bar{z}z}D_{\bar{z}}\lambda_z - 2\zeta^{\bar{z}z}D_z\lambda_{\bar{z}} + 2\zeta^{\bar{z}z}D_{\bar{z}}\lambda_z \\
&\qquad\qquad - 2\alpha^{\bar{z}z}[\sigma_z,\lambda_{\bar{z}}] - 2\alpha^{\bar{z}z}[\sigma_{\bar{z}},\lambda_z] + 2\zeta^{\bar{z}z}[\sigma_z,\lambda_{\bar{z}}] + 2\zeta^{\bar{z}z}[\sigma_{\bar{z}},\lambda_z] \\
&\qquad\qquad + 2\alpha^{\bar{z}z}D_z(V^z\zeta_{z\bar{z}}) - 2\alpha^{\bar{z}z}D_{\bar{z}}(V^{\bar{z}}\zeta_{z\bar{z}}) - \alpha^{\bar{z}z}\mathcal{L}_V\alpha_{z\bar{z}} + \alpha^{\bar{z}z}[\iota_V\sigma,\alpha_{z\bar{z}}] \\
&\qquad\qquad - \zeta^{\bar{z}z}\mathcal{L}_V\zeta_{z\bar{z}} - \zeta^{\bar{z}z}[\iota_V\sigma,\zeta_{z\bar{z}}]\big).
\end{aligned} \tag{4.15}
$$

For the chiral multiplet, we define the action by

$$
\begin{aligned}
S_{\mathrm{C}} &= \int_\Sigma \delta\,\mathrm{Tr}\left(\rho\wedge\star\overline{\delta\rho} - \phi\bar{\phi}\alpha + \mathtt{F}\wedge\star\bar{\mu} + 2\sigma\phi\wedge\star\iota_V\bar{\mu}\right) \\
&= \int_{\Sigma^2} \mathrm{Tr}\left((\mathrm{d}_A\phi + \iota_V\mathtt{F})\wedge\star(\mathrm{d}_A\bar{\phi} + \iota_V\bar{\mathtt{F}}) + \sigma\phi\wedge\star\bar{\phi}\sigma + \mathrm{i}\phi\bar{\phi}\mathtt{D}' + \mathtt{F}\wedge\star\bar{\mathtt{F}}\right. \\
&\qquad \left. - \iota_V\mathtt{F}\wedge\star(\bar{\phi}\sigma) + \sigma\phi\wedge\star\iota_V\bar{\mathtt{F}} - \partial(\phi\bar{\phi}\sigma) - \bar{\partial}(\phi\bar{\phi}\sigma)\right) + S_{\mathrm{C}f}.
\end{aligned} \tag{4.16}
$$

Note that $-\iota_V\mathtt{F}\wedge\star(\bar{\phi}\sigma) + \sigma\phi\wedge\star\iota_V\bar{\mathtt{F}}$ is purely imaginary. Here the fermionic part reads

$$
\begin{aligned}
S_{\mathrm{C}f} = \int_\Sigma \mathrm{Tr}\big(&-\rho\wedge\star\mathrm{d}_A\bar{\eta} + \mathrm{d}_A\rho\star\bar{\mu} \\
&+ \rho\wedge\star(\bar{\eta}\sigma) - \sigma\wedge\rho\star\bar{\mu} \\
&- 2\rho\wedge\star(\lambda\bar{\phi}) - \phi\bar{\eta}\alpha + \zeta\phi\star\bar{\mu} \\
&- \rho\wedge\star\iota_V(\mathrm{d}_A\iota_V\bar{\mu}) \\
&+ 2\sigma\iota_V\rho\wedge\star\iota_V\bar{\mu} - \rho\wedge\star\iota_V(\bar{\mu}\iota_V\sigma) \\
&+ \rho\wedge\star\bar{\phi}\iota_V\zeta - \iota_V\rho\bar{\phi}\alpha + 2\lambda\phi\wedge\star\iota_V\bar{\mu} + 2\iota_V\zeta\phi\wedge\star\iota_V\bar{\mu}\big).
\end{aligned}
$$

As for the Q-invariant superpotential term, it takes the standard form:

$$
S_{\mathrm{W}} = \mathrm{i}\int_\Sigma \frac{\partial W}{\partial\phi}\mathtt{F} + \frac{1}{2}\frac{\partial^2 W}{\partial\phi^2}\rho\wedge\rho + \delta(\bar{\mu}\partial_{\bar{\phi}}\bar{W}), \tag{4.17}
$$

where the superpotentials W and $\bar{W}$ are functions of ϕ and $\bar{\phi}$ respectively. It is easy to see that S_{W} is Q-invariant up to a total derivative. However, such a total derivative term can not be ignored when Σ has a boundary. In such a case, we also have to ask whether the supersymmetry invariance for S_{C} and S_{V} would be broken by such a boundary. This luckily would not happen, as it is easy to realize that the Q-exact terms S_{C} and S_{V} remain Q-invariant:

$$\int_\Sigma [Q, \{Q, \mathcal{V}\}] = \int_\Sigma (\mathrm{d}\iota_V + \iota_V \mathrm{d})\mathcal{V} = \int_{\partial\Sigma} \iota_V \mathcal{V} = 0. \tag{4.18}$$

So the trouble only arises with S_{W}. Computing its Q-variation, we end up with the non-vanishing boundary term:

$$[Q, S_W] = \mathrm{i} \int_\Sigma \mathrm{d}(\rho \partial_\phi W) = \mathrm{i} \int_{\partial\Sigma} \rho \partial_\phi W. \tag{4.19}$$

Then how to make the theory Q-invariant? One way is to set $\partial_\phi W{=}0$ on the boundary as B-twisted Landau-Ginsberg models do. In our Ω-deformed model, there is another way by adding a boundary action which cancels the Q-variation of S_W, which is shown as follows.

In the polar coordinate system, the Ω-deformed theory has rotation symmetry; and we can take $V = \varepsilon \partial_\varphi$, with ε constant. In this situation, we can revive the Q-invariant action by adding the following boundary action:

$$-\frac{\mathrm{i}}{\varepsilon} \int_{\partial\Sigma} \mathrm{d}\varphi (W + W_0), \tag{4.20}$$

where W_0 is a locally constant independent of ϕ. This term is interesting by itself since it can only emerge when we turn on the Ω-deformation. Here the only condition is that this term must be bounded below by an constant, which requires

$$\frac{1}{\varepsilon} \mathrm{Im}(W + W_0) \geq 0. \tag{4.21}$$

Note that in this boundary action term, the Ω-deformation parameter ε plays the role of the Planck constant. Thus when we take the classical limit $\varepsilon \to 0$, the field configurations should obey the equation of motion $\partial_\phi W = 0$ on the boundary, and we restore the B-brane constraint.

4.2.2 Exploring the theory: localization on the Higgs branch

As promised in the introduction, we have succeeded in constructing the Ω-deformed B-twisted gauge theory. In this section let us further explore the physical properties of this theory.

The first property is the (quasi-)topological invariance. Since the metric-dependent terms S_{V} and S_{C} in the action are Q-exact, the variation of the action with respect to the metric is Q-exact. This leads to our theory being topologically invariant under the metric variation, *so long as the vector V remains Killing.*

Second, to understand the theory more deeply, we resort to the localization method. By rescaling the Q-exact terms by a very large factor t, we localize the path integral to the locus where the bosonic terms in S_{V} and S_{C} all vanish. Let us elaborate it as follows.

Here we consider the theory on the world sheet $\mathbb{D}^2$ with a boundary. We consider the boundary topologically as a circle, and the Killing vector field V generates its rotations. The neighborhood of the boundary looks like a short cylinder with coordinates (n, φ), where n parametrizes the direction normal to the boundary. We equip this cylinder with a flat metric. After the boundary condition is fixed, one can deform the metric to anything that is allowed by the quasi-topological property of the theory.

We first analyze boundary conditions for the vector multiplet fields. The gauge field has the standard kinetic term, so its boundary conditions is a standard one, namely either the Dirichlet or Neumann boundary condition. Since a gauge-invariant expression for the former condition does not exist in two dimensions, we choose the latter, $F_{n\varphi} = 0$. Gauging A_n away, we can write this condition as $\partial_n A_\varphi = 0$. The requirement of Q-invariance then leads to $\partial_n \sigma_\varphi = \lambda_n = \partial_n \lambda_\varphi = 0$. If we now look at the kinetic terms for σ in the vector multiplet action 4.13, we notice that it differs from the standard one by total derivative terms. A natural way to kill these unwanted terms is to set $\sigma_n = 0$ on the boundary; the total derivative terms in the chiral multiplet action 4.16 also drop out then. Taking the Q-variation of this condition, we get $\zeta_{n\varphi} = 0$. In fact, the set of boundary conditions we have found so far is part of the conditions imposed by a

B-brane in $N=(2,2)$ gauge theory [71, 72]. This suggests that we should choose our boundary condition for the vector multiplet to be the B-brane boundary condition:

$$A_n = \partial_n A_\varphi = \sigma_n = \partial_n \sigma_\varphi = \lambda_n = \partial_n \lambda_\varphi = \zeta_{n\varphi} = \partial_n \alpha_{n\varphi} = \partial_n \mathtt{D}_{n\varphi} = 0. \quad (4.22)$$

This set of boundary conditions is not Q-invariant by itself. In order to achieve Q-invariance, we further impose an infinite series of conditions, generated from the above conditions by the action of even powers of ∂_n [71].

To perform the localization, we rescale the Q-exact terms by

$$t\mathcal{S}_{\mathrm{V}} + t\mathcal{S}_{\mathrm{C}} \quad (4.23)$$

Among the Q-exact terms, in addition, we rescale $\delta\,\mathrm{Tr}(\mathtt{F} \wedge \star\bar{\mu})$ by an additional factor s as

$$st\delta\,\mathrm{Tr}(\mathtt{F} \wedge \star\bar{\mu}). \quad (4.24)$$

Here we take $t \to \infty$, $s \to \infty$ to perform the localization.

First, since we rescale the Q-exact term $\delta(\mathtt{F} \wedge \star\bar{\mu})$ by st instead of just t, it is easy to see that the auxiliary field $\mathtt{F}$ will be forced to take the locus $\mathtt{F} = 0$. So any term containing $\mathtt{F}$ vanishes in the saddle-point configurations. Given this, after integrating out the other auxiliary field $\mathtt{D}$, the saddle-point configurations for the rest of the bosons read:

$$F_{\mu\nu} = D_\mu \sigma_\nu = D_\mu \phi = [\sigma, \sigma] = \mathcal{R}\sigma^2 = [\sigma, \phi] = [\phi, \bar{\phi}] = 0. \quad (4.25)$$

Since $\mathbb{D}^2$ is simply connected, $F_{\mu\nu} = 0$ gives us $A_\mu = 0$ up to gauge transformations. And by the quasi-topological property of our theory, we can always deform the disk so that $\mathcal{R}$ is non-zero, which forces the locus of σ to be zero. Therefore the vacua reduce to the Higgs branch. Accordingly, the saddle-point configurations for ϕ reduce to

$$\partial_\mu \phi = 0, \;\; [\phi, \bar{\phi}] = 0. \quad (4.26)$$

To perform the path integral computation, we have yet to fix the boundary conditions for the chiral multiplet. The guiding principle to determine the boundary conditions is the following. The action must be invariant under variations of fields up to the bulk equations of motion. That is, $\delta S = 0$ with respect to relevant fields, modulo the bulk equations of motion. It follows that the equations constraining the fields on the boundary emerge.

Guided by this principle, after integrating out the auxiliary fields, provided the equations of motion satisfied, we obtain the following constraints on the boundary:

$$\begin{aligned} &\delta\bar{\phi}\partial_n\phi + \delta\phi\partial_n\bar{\phi} = 0, \\ &\delta\rho_\varphi\bar{\mu}_{\varphi n} + \delta\bar{\eta}\rho_n = 0. \end{aligned} \tag{4.27}$$

Given that the equations of (4.27) are satisfied, as well as given that these equations themselves are invariant under supersymmetry transformations, we have the freedom to choose the boundary conditions. Different choices can lead to different physical results.

For our current purpose, without discussing the boundary condition in more detail, let us jump to the final result. Let us look at the partition function Z. It is easy to see that after the localization is done, we should obtain

$$Z = \int \mathrm{d}\phi_0 Z_1(\phi_0)\exp\left(-\frac{\mathrm{i}}{\varepsilon}\int_{\partial\Sigma}(W(\phi_0) + W_0)\right), \tag{4.28}$$

where ϕ_0 denotes the saddle-point configurations of ϕ that satisfy the boundary conditions, and $Z_1(\phi_0)$ is obtained by integrating out all the fluctuations of the fields around the saddle-point configurations and all the fermion zero modes (if they exist) in the path integral where the boundary conditions are satisfied. Note that the boundary term must be bounded below by an constant: $\frac{1}{\varepsilon}\mathrm{Im}(W+W_0) \geq 0$.

As indicated by this result, our model has interesting applications, one can have an idea of this through the following discussion. In the case considered here we only have the Higgs branch vacua, thus modulo the contributions of the vector multiplet in Z_1, our theory reduces to a sigma model, which can be considered as the Ω-deformation of the B-twisted Landau-Ginzburg model constructed in [73], with the target space flat. Therefore, if we consider the boundary conditions for the chiral multiplet to be the same as that considered, our model can have applications similar to those elaborated in that paper.

Last but not least, as mentioned in the introduction, in the next section, this model will show another important and interesting application in the 3d/3d correspondence. We will use this 2d model as a tool to construct a 5d theory by "lifting". Starting with this 5d theory we can obtain the complex CS theory which sits on one side of the 3d/3d correspondence.

4.3 3d Complex CS from 5d SYM

As explained in the introduction, the main mission of this chapter is to show that by localization, a 5d SYM on $\mathbb{D}^2_\varepsilon \times M$ can reduce to the complex Chern-Simons theory described in the 3d/3d correspondence. In this section we are going to accomplish it.

4.3.1 5d SYM on $\mathbb{D}^2_\varepsilon \times M$

We first construct the 5d SYM on $\mathbb{D}^2_\varepsilon \times M$ with gauge group $\mathcal{G}$, where the disk $\mathbb{D}^2$ is equipped with metric h and the three-manifold M is equipped with metric g_M. The metric of the total product space is $g = h \oplus g_M$. The theory is topologically twisted along M and possesses the Ω-deformed B-twisted supercharge on $\mathbb{D}^2$.

As was said, we use the 2d Ω-deformed B-twisted theory formulated in the previous section as a tool to construct such a 5d theory. We obtain the transformation rules and the action by lifting their counterparts in the 2d theory to 5d.

To achieve this goal, first of all, we need to understand how the field contents of the 5d theory decompose as supermultiplets of the 2d theory. Let us elucidate this decomposition as follows.

It is well known that 5d $N = 2$ SYM can be obtained via dimensional reduction from N=1 MSYM in ten dimensions, which has ten gauge fields and sixteen fermions. Assume that the 10d theory is formulated on a spin-ten manifold T, for which the structure group of the spin bundle is $Spin(10)_T$. After the dimensional reduction from T to a generic spin-five manifold Y, $Spin(10)_T$ is broken to

$$Spin(5)_Y \times Spin(5)_R, \tag{4.29}$$

where $Spin(5)_Y$ is the structure group for Y and $Spin(5)_R$ denotes the R-symmetry. Under the symmetry (4.29), the fields transform as follows:

$$\begin{aligned} A&\colon (\mathbf{5},\mathbf{1}),\\ X&\colon (\mathbf{1},\mathbf{5}),\\ \Psi&\colon (\mathbf{4},\mathbf{4}), \end{aligned} \tag{4.30}$$

where A, X and Ψ are the gauge field, Higgs fields and fermions in the 5d theory respectively. For the flat case $Y=\mathbb{R}^5$, the theory can preserve sixteen supercharges transforming as $(\mathbf{4},\mathbf{4})$.

For the case of $Y=\mathbb{D}^2\times M$ we are interested in, in general, the supersymmetries are completely broken. But by a partial twisting on Y, we can preserve a fraction of them.

We perform such a twist in the following way. On $Y=\mathbb{D}^2\times M$, the structure group $Spin(5)_Y$ breaks to

$$Spin(2)_{\mathbb{D}^2}\times Spin(3)_M \cong U(1)_{\mathbb{D}^2}\times SU(2)_M. \tag{4.31}$$

To perform the partial twist, we first split the R-symmetry group $Spin(5)_R$ to

$$Spin(2)_R\times Spin(3)_R \cong U(1)_R\times SU(2)_R. \tag{4.32}$$

With this in hand, the partial twisting is done by defining a new symmetry $SU(2)'_M$ by the diagonal subgroup of $SU(2)_M\times SU(2)_R$. Then the full symmetry group of the twisted theory is $SU(2)'_M\times U(1)_{\mathbb{D}^2}\times U(1)_R$, under which the fields transform as:

$$\begin{aligned} A &: 1^{(\pm2,0)}\oplus 3^{(0,0)},\\ X &: 1^{(0,\pm2)}\oplus 3^{(0,0)},\\ \Psi &: 1^{(\pm1,\pm1)}\oplus 3^{(\pm1,\pm1)}. \end{aligned} \tag{4.33}$$

In the similar manner to the fermions, it is clear that after the twisted four supercharges become scalars on M and thus can be preserved on Y. From the two-dimensional viewpoint, these supercharges generate the $\mathcal{N} = (2,2)$ supersymmetry on $\mathbb{D}^2$.

To obtain the desired Ω-deformation of the B-twisted supercharges on $\mathbb{D}^2$, we further perform the twisting along $\mathbb{D}^2$ by defining a new $U(1)'_{\mathbb{D}^2}$ by the diagonal subgroup of $U(1)_{\mathbb{D}^2} \times U(1)_R$. After doing this, the full symmetry group of the theory becomes $SU(2)'_M \times U(1)'_{\mathbb{D}^2}$, under which, the fields transform as:

$$\begin{aligned}
&A : 1^{\pm 2} \oplus 3^0, \\
&X : 1^{\pm 2} \oplus 3^0, \\
&\Psi : 2 \times 1^0 \oplus 1^{\pm 2} \oplus 2 \times 3^0 \oplus 3^{\pm 2}.
\end{aligned} \tag{4.34}$$

Here one can find that the fields split into two parts: scalars on M and one-forms on M.

For the first part, boson scalars on M are all one-forms on $\mathbb{D}^2$; two of the fermion scalars on M are one-forms, while the other two are scalars on $\mathbb{D}^2$. These scalar fields on M form a vector multiplet on $\mathbb{D}^2$, supplemented with a real auxiliary field $\mathtt{D}$:

$$(A_i, \sigma_i, \lambda_i, \chi, \bar{\chi}, \mathtt{D}), \tag{4.35}$$

where i denotes the coordinate index on $\mathbb{D}^2$. In constrast, we use m to denote the coordinate index on M.

As for the second part, the gauge field components A_m and the remaining Higgs fields X_m on M can be combined to define the following new complex gauge fields:

$$\mathcal{A}_m = A_m + \mathrm{i} X_m, \ \bar{\mathcal{A}}_m = A_m - \mathrm{i} X_m. \tag{4.36}$$

The remaining one-forms on M are fermions: $3^{\pm 2}$ are one-forms on $\mathbb{D}^2$; 2×3^0 are scalars on $\mathbb{D}$. From the two-dimensional viewpoint, supplemented with auxiliary fields $\mathtt{F}_m$ and $\bar{\mathtt{F}}_m$, these one-forms on M form the chiral multiplet on $\mathbb{D}^2$:

$$(\mathcal{A}_m, \bar{\mathcal{A}}_m, \rho_{im}, \eta_m, \bar{\eta}_m, \mathtt{F}_m, \bar{\mathtt{F}}_m). \tag{4.37}$$

The B-twisted vector and chiral multiplets in 2d are summarized in section 2. Remember, to construct the Ω-deformation of B-twisted gauge theory on $\mathbb{D}^2$, *some of the component fields are redefined*, but it is clear that this decomposition relation between the component fields of the 5d and the 2d theories holds under the redefinitions. The following diagram summarizes this relation:

5d	A_i	X_i	λ_i	ζ_{ij}	α_{ij}	$\mathtt{D}_{ij}$	$A_m + \mathrm{i}X_m$	$A_m - \mathrm{i}X_m$	ρ_{im}	$\bar{\eta}_m$	$\bar{\mu}_{ijm}$	$\mathtt{F}_{ijm}$	$\bar{\mathtt{F}}_{ijm}$
2d	A_i	σ_i	λ_i	ζ_{ij}	α_{ij}	$\mathtt{D}_{ij}$	ϕ	$\bar{\phi}$	ρ_i	$\bar{\eta}$	$\bar{\mu}_{ij}$	$\mathtt{F}_{ij}$	$\bar{\mathtt{F}}_{ij}$

With this clear, now we go on constructing the supersymmetry transformation laws and action for the 5d SYM on $\mathbb{D}^2_\varepsilon \times M$ by directly lifting their counterparts of the 2d theory constructed in section 2. Let us reveal the construction in the following.

4.3.1.1 Supersymmetry transformations

For those decomposing to the 2d vector multiplet, the 5d component fields are all scalars on M. Thus the lifting merely adds to the 2d fields the coordinate-dependence on M. The form of the transformation laws for the lifted 5d theory therefore holds:

$$
\begin{aligned}
\delta A_i &= \mathrm{i}\lambda_i,\\
\delta X_i &= \lambda_i + (\iota_V \zeta)_i,\\
\delta \lambda_i &= -\mathrm{i}(\iota_V F)_i + D_i(\iota_V X),\\
\delta \zeta_{ij} &= \mathrm{i}F_{ij} + (\mathrm{d}_A X)_{ij} - (X \wedge X)_{ij},\\
\delta \alpha_{ij} &= (\mathrm{d}_A X)_{ij} + \mathrm{i}\mathtt{D}_{ij},\\
\delta \mathtt{D}_{ij} &= -\mathrm{i}(\mathrm{d}_A \iota_V \alpha)_{ij} + \mathrm{i}[\iota_V X, \alpha_{ij}] + \mathrm{i}(\mathrm{d}_A \lambda)_{ij} + \mathrm{i}[X_i, \lambda_j] + \mathrm{i}(\mathrm{d}_A \iota_V \zeta)_{ij}.
\end{aligned}
\tag{4.38}
$$

For which decomposing to the 2d chiral multiplet, the 5d component fields are one-forms on M. Thus besides adding to the coordinate-dependence on M, the lifting should also give them the coordinate index of M. In addition, to restore the differential operators along M, we perform the following promotion:

$$
\begin{aligned}
\mathcal{A}_m &\to \mathrm{i}\mathcal{D}_m, \quad [\mathrm{i}\mathcal{D}_m, \mathrm{i}\mathcal{D}_n] \to \mathrm{i}\mathcal{F}_{mn},\\
\bar{\mathcal{A}}_m &\to \mathrm{i}\bar{\mathcal{D}}_m, \quad [\mathrm{i}\bar{\mathcal{D}}_m, \mathrm{i}\bar{\mathcal{D}}_n] \to \mathrm{i}\bar{\mathcal{F}}_{mn},
\end{aligned}
\tag{4.39}
$$

where $\mathcal{D}_m = \nabla_m - \mathrm{i}\mathcal{A}_m$ and $\bar{\mathcal{D}}_m = \nabla_m - \mathrm{i}\bar{\mathcal{A}}_m$, with ∇_m the covariant derivatives and $\mathcal{A}_m = A_m + \mathrm{i}X_m$ the complex gauge fields lifted from ϕ. These derivatives can provide kinetic terms along M in the action. The lifted supersymmetry transformations thus read:

$$
\begin{aligned}
\delta\mathcal{A}_m &= (\iota_V\rho)_m, \\
\delta\bar{\mathcal{A}}_m &= \bar{\eta}_m, \\
\delta\rho_{im} &= \mathcal{F}_{im} + \mathrm{i}\mathcal{D}_m X_i + (\iota_V \mathtt{F})_{im}, \\
\delta\bar{\eta}_m &= (\iota_V\bar{\mathcal{F}})_m + \mathrm{i}\bar{\mathcal{D}}_m(\iota_V X), \\
\delta\bar{\mu}_{ijm} &= \bar{\mathtt{F}}_{ijm}, \\
\delta\mathtt{F}_{ijm} &= (\mathrm{d}_A\rho)_{ijm} - (X\wedge\rho)_{ijm} - \mathrm{i}\mathcal{D}_m\zeta_{ij}, \\
\delta\bar{\mathtt{F}}_{ijm} &= (\mathrm{d}_A\iota_V\bar{\mu})_{12m} + [\bar{\mu}_{ijm},\ \iota_V X],
\end{aligned}
\tag{4.40}
$$

where $\mathcal{F}_{im} = \mathrm{i}[D_i, \mathcal{D}_m]$ and $\bar{\mathcal{F}}_{im} = \mathrm{i}[D_i, \bar{\mathcal{D}}_m]$.

For (4.38) and (4.40), straightforwardly, one can check that the supersymmetry algebra is closed up to the Lie derivative by V and gauge transformations:

$$\delta^2 = \mathcal{L}_V - G(V^i X_i). \tag{4.41}$$

$$G(V^i X_i)A_M = -\mathrm{i}\mathrm{d}_{A_M}(V^i X_i),\ G(V^i X_i)\Phi = [V^i X_i, \Phi], \tag{4.42}$$

where A denotes the gauge field, while Φ denote the other, and M denotes the coordinate index for the whole 5d space.

We thus successfully constructed the supersymmetry transformation laws of the desired 5d theory, where the supercharge be topologically twisted along M and be the Ω-deformation of the B-twisted supercharges on $\mathbb{D}^2$.

4.3.1.2 Action

Next, we proceed to construct the action of the 5d theory on $\mathbb{D}^2 \times M$. Remember that the Q-invariant action of our 2d theory on $\mathbb{D}^2_\varepsilon$ consists of the following parts:

$$S = S_{\mathrm{V}} + S_{\mathrm{C}} + S_{\mathrm{W}} + S_{\mathrm{boundary}}. \tag{4.43}$$

The first three are the actions for the vector multiplet, the chiral multiplet and the superpotential respectively; the last is the boundary term necessarily added to make the theory supersymmetric invariant on $\mathbb{D}^2$ with a boundary. By lifting them to the 5d, we write down the action of our 5d theory as follows.

The first part is lifted to

$$\begin{aligned}\mathcal{S}_{\mathrm{V}}^{5d} &= \int_M \int_{\mathbb{D}^2} \delta\,\mathrm{Tr}\big(\alpha \wedge \star(-\mathrm{d}_A\sigma + \mathrm{i}\mathtt{D} + 4\bar{D}\sigma) + \zeta \wedge \star\overline{\delta\zeta}\big) \\ &= \int \sqrt{g}\mathrm{d}^5x\ \mathrm{Tr}\big(\frac{1}{2}F_{ij}^2 + D_iX_jD^iX^j - \frac{1}{2}[X_i, X_j]^2 + \mathcal{R}X_iX^i + \mathtt{D}'_{ij}\mathtt{D}'^{ij} \\ &\qquad + 2\mathrm{id}(X_i\mathrm{d}_AX^i) + 2\epsilon^{ij}\mathrm{d}(X_i\mathrm{d}_AX_j)\big) + \mathcal{S}_{\mathrm{V}f}^{5d},\end{aligned} \tag{4.44}$$

where the fermionic term

$$\begin{aligned}\mathcal{S}_{\mathrm{V}f}^{5d} &= -\int_M \int_{\mathbb{D}^2} \mathrm{Tr}\left(\alpha \wedge \star\delta(-\mathrm{d}_A\sigma + \mathrm{i}\mathtt{D} + 4\bar{D}\sigma) + \zeta \wedge \star\delta\overline{\delta\zeta}\right) \\ &= -\int_M \sqrt{g_M}\mathrm{d}^3x \int_{\mathbb{D}^2} \mathrm{Tr}(2\alpha D_i\lambda^i + 2\epsilon^{ij}\zeta D_i\lambda_j \\ &\qquad + 2\epsilon^{ij}\alpha\lambda_iX_j + 2\epsilon^{ij}\zeta\lambda_jX_j \\ &\qquad + 2\alpha D_i(\iota_V\zeta)^i - \alpha D_i(\iota_V\alpha)^i + \epsilon^{ij}\alpha[\iota_VX, \alpha_{ij}] \\ &\qquad + \epsilon^{ij}\zeta D_i(\iota_V\zeta)_j + \epsilon^{ij}\zeta[(\iota_V\zeta)_i, X_j]\big).\end{aligned} \tag{4.45}$$

The second part is lifted to

$$\begin{aligned}\mathcal{S}_{\mathrm{C}}^{5d} &= \int_M \int_{\mathbb{D}^2} \delta\,\mathrm{Tr}\left(\rho \wedge \star\overline{\delta\rho} + \mathrm{d}_{\mathcal{A}} \wedge \star\mathrm{d}_{\bar{\mathcal{A}}}\alpha + \mathtt{F} \wedge \star\bar{\mu} + 2X \wedge \mathrm{d}_{\mathcal{A}} \wedge \star\iota_V\bar{\mu}\right) \\ &= \int \sqrt{g}\mathrm{d}^5x\mathrm{Tr}\big((\mathcal{F} + \iota_V\mathtt{F})_{im}(\bar{\mathcal{F}} + \iota_V\bar{\mathtt{F}})^{im} + \mathcal{D}_mX_i\bar{\mathcal{D}}^mX^i - \mathrm{i}[\mathcal{D}_m, \bar{\mathcal{D}}^m]\mathtt{D}' + \mathtt{F}_{ijm}\bar{\mathtt{F}}^{ijm} \\ &\qquad - \mathrm{i}(\iota_V\mathtt{F})_{im}(\bar{\mathcal{D}}^mX^i) - \mathrm{i}(\mathcal{D}_mX_i)(\iota_V\bar{\mathtt{F}})^{im} + \partial_i([\mathcal{D}_m, \bar{\mathcal{D}}^m]X^i)\big) + \mathcal{S}_{\mathrm{C}f}^{5d} \\ &= \int \sqrt{g}\mathrm{d}^5x\mathrm{Tr}\big(F_{im}F^{im} + D_iX_mD^iX^m + D_mX_iD^mX^i \\ &\qquad - [X_m, X_i]^2 + 2\mathrm{i}D_mX^m\mathtt{D}' + \mathtt{F}_{ijm}\bar{\mathtt{F}}^{ijm} \\ &\qquad + (\iota_V\mathtt{F})_{im}(F^{im} - \mathrm{i}D^iX^m) + (F_{im} + \mathrm{i}D_iX_m)(\iota_V\bar{\mathtt{F}})^{im} \\ &\qquad - \mathrm{i}(\iota_V\mathtt{F})_{im}(D^m - X^m)X^i - \mathrm{i}(D_m + X_m)X_i(\iota_V\bar{\mathtt{F}})^{im} \\ &\qquad + (\iota_V\mathtt{F})_{im}(\iota_V\bar{\mathtt{F}})^{im} - 2\partial_i(D_mX^mX^i)\big) + \mathcal{S}_{\mathrm{C}f}^{5d},\end{aligned} \tag{4.46}$$

where the fermionic term

$$\begin{aligned}\mathcal{S}^{5d}_{\mathrm{C}f} = \int \sqrt{g}\mathrm{d}^5x\,\mathrm{Tr}\big(& -\rho_{im}D^i\bar{\eta}^m + \epsilon^{ij}D_i\rho_{jm}\bar{\mu}^m \\ & + \rho_{im}\bar{\eta}^m X^i - X^i\rho_{im}\bar{\mu}^m \\ & - 2\mathrm{i}\rho_{im}\bar{\mathcal{D}}^m\lambda^i - \mathrm{i}\epsilon^{ij}\mathcal{D}_m\bar{\eta}^m\alpha_{ij} + \mathrm{i}\epsilon^{ij}\zeta_{ij}\mathcal{D}_m\bar{\mu}^m \\ & + \rho_{im}V_j(D_kV^k\bar{\mu}^{ijm}) \\ & - 2X_iV^j\rho_{jm}V_k\bar{\mu}^{ikm} + \rho_{im}V_k(\bar{\mu}^{ikm}V^jX_j) \\ & - \mathrm{i}\rho_{im}\bar{\mathcal{D}}^mV_j\zeta^{ij} - \mathrm{i}\epsilon^{ij}V^i\rho_{im}\bar{\mathcal{D}}^m\alpha_{ij} - 2\mathrm{i}\lambda_i\mathcal{D}_mV_j\bar{\mu}^{ijm} + 2\mathrm{i}V^j\zeta_{ij}\mathcal{D}_mV_j\bar{\mu}^{ijm}\big).\end{aligned}$$

Here $\star$ denotes the Hodge star operator on $\mathbb{D}^2 \times M$.

To construct the 5d SYM action containing full kinetic terms, it turns out that we should make the lifted superpotential W take the form of the Chern-Simons functional on the 3-manifold M:

$$W = \frac{1}{2}\mathrm{Tr}\int_M \mathrm{CS}(\mathcal{A}) = \frac{1}{2}\int_M \mathrm{Tr}(\mathcal{A}\wedge\mathrm{d}\mathcal{A} - \frac{2\mathrm{i}}{3}\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}), \tag{4.47}$$

which leads to

$$\mathcal{S}^{5d}_{\mathrm{W}} = \mathrm{i}\int_{\mathbb{D}^2\times M}\mathrm{Tr}(\mathtt{F}\wedge\mathcal{F} + \frac{1}{2}\rho\wedge\mathrm{d}_{\mathcal{A}}\rho + \bar{\mathtt{F}}\wedge\bar{\mathcal{F}} - \frac{1}{2}\bar{\mu}\wedge\mathrm{d}_{\bar{\mathcal{A}}}\bar{\eta}), \tag{4.48}$$

where $\mathrm{d}_{\mathcal{A}_m} = \mathrm{d}_m - \mathrm{i}\mathcal{A}_m$ and $\mathcal{F} = \mathrm{d}\mathcal{A} - \mathrm{i}\mathcal{A}\wedge\mathcal{A}$.

It is clear that in order to make the 5d theory Q-invariant on $\mathbb{D}^2 \times M$ when $\mathbb{D}^2$ has a boundary, we also need to include in the action the following lifted boundary term:

$$\begin{aligned}\mathcal{S}^{5d}_{\mathrm{boundary}} &= -\frac{\mathrm{i}}{\varepsilon}\int_{\partial\mathbb{D}^2}\mathrm{d}\varphi(W + W_0) \\ &= -\frac{\mathrm{i}}{\varepsilon}\int_{\partial\mathbb{D}^2}\mathrm{d}\varphi\big(\frac{1}{2}\int_M \mathrm{Tr}(\mathcal{A}\wedge\mathrm{d}\mathcal{A} - \frac{2\mathrm{i}}{3}\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}) + W_0\big).\end{aligned} \tag{4.49}$$

One can see that the lifting operation commutes with any derivation, especially supersymmetry transformation, that commutes with ∇. Therefore it follows that the lifted supersymmetric action is invariant under the lifted supersymmetry. The full action of our 5d theory on $\mathbb{D}^2 \times M$ is:

$$\mathcal{S}^{5d} = \mathcal{S}^{5d}_{\mathrm{V}} + \mathcal{S}^{5d}_{\mathrm{C}} + \mathcal{S}^{5d}_{\mathrm{W}} + \mathcal{S}^{5d}_{\mathrm{boundary}}. \tag{4.50}$$

Thus the construction of the 5d theory is successfully done.

Finally, to see that our choice of the form of W and our action constructed by lifting are sensible, we can take $M = \mathbb{R}^3$, and replace $\mathbb{D}^2$ with $\mathbb{R}^2$. Given this, if we turn off the Ω-deformation, after integrating out the auxiliary fields $\mathtt{D}$ and $\mathtt{F}$, the bosonic part of the 5d action reduces to

$$\begin{aligned}
\mathcal{S}^{5d}_b =& \mathcal{S}^{5d}_{\mathrm{C}b} + \mathcal{S}^{5d}_{\mathrm{V}b} + \mathcal{S}^{5d}_{\mathrm{W}b} \\
=& \int \sqrt{g}\mathrm{d}^5 x \ \mathrm{Tr}\big(\frac{1}{2}F_{ij}^2 + D_i X_j D^i X^j - \frac{1}{2}[X_i, X_j]^2 \\
& \qquad + F_{im}F^{im} + D_i X_m D^i X^m + D_m X_i D^m X^i - [X_m, X_i]^2 \\
& \qquad + \frac{1}{2}F_{mn}F^{mn} + D_m X_n D^m X^n - \frac{1}{2}[X_m, X_n]^2\big) \\
=& \int \sqrt{g}\mathrm{d}^5 x \ \mathrm{Tr}\big(\frac{1}{2}F_{MN}F^{MN} + D_M X_N D^M X^N - \frac{1}{2}[X_M, X_N]^2\big),
\end{aligned} \tag{4.51}$$

which is precisely the bosonic part of the 5d MSYM action on $\mathbb{R}^5$.

4.3.2 Localization to M

Since we already have the action of the 5d theory, it is time to carry out our promise made in the introduction: to derive the 3d Chern-Simons theory with complex group $\mathcal{G}_{\mathbb{C}}$ by localization. It is fulfilled in this section.

4.3.2.1 Boundary conditions

Before performing the localization of the 5d theory on $\mathbb{D}^2$, we have to fix the boundary conditions of the field variables. Let us address this issue in the following.

Firstly, for the field components which are lifted from the vector multiplet of the 2d theory, following the analysis in 4.2.2 we obtain the boundary conditions as

$$A_n = \partial_n A_\varphi = X_n = \partial_n X_\varphi = \lambda_n = \partial_n \lambda_\varphi = \zeta_{n\varphi} = \partial_n \alpha_{n\varphi} = \partial_n \mathtt{D}_{n\varphi} = 0. \quad (4.52)$$

Here we consider the boundary of $\mathbb{D}^2$ topologically as a circle, and the neighborhood of the boundary looks like a short cylinder with coordinates (n, φ), with n parametrizing the direction normal to the boundary. We equip this cylinder with a flat metric, and the Killing vector field V generates the circle's rotations. After the boundary condition is fixed, one can deform the metric to anything that is allowed by the quasi-topological property of the theory. Note that in order to achieve Q-invariance for the boundary condition, we further impose an infinite series of conditions, generated from the above conditions by the action of even powers of ∂_n [71].

With this in hand, let us go on discussing the boundary conditions for the rest of the fields.

The rest of the fields are lifted from the chiral multiplet of the 2d Ω-deformed B-twisted theory. For the chiral multiplet of the B-twisted Landau-Ginzburg models in the presence of the Ω-deformation [73], the boundary conditions are analogous to the boundary conditions in the non-Ω-deformed A-twisted theory, i.e. the A-brane boundary conditions.

Motivated by this observation, we are going to impose on the rest of the fields the boundary conditions which are (from the 2d theory viewpoint) close to the A-brane boundary conditions. For our mission of obtaining the complex Chern-Simons, this will turn out to be the suitable choice, as will be shown.

We use $\mathcal{Y}$ to denote the complex space on which the complex gauge fields $(\mathcal{A}_m, \bar{\mathcal{A}}_m)$ take values. On the boundary of $\mathbb{D}^2$, we require

$$(\mathcal{A}_m, \bar{\mathcal{A}}_m) \in \gamma, \quad (4.53)$$

where γ is set to be the Lagrangian submanifold of $\mathcal{Y}$ with respect to the Kähler form

$$\omega = \int_M \mathrm{Tr}(\delta A \wedge \star \delta X), \quad (4.54)$$

where δ denotes the exterior differential in this space on γ.

The field variables are also constrained as follows. The action must be invariant under variations of fields up to the bulk equations of motion. That is, $\delta S = 0$ with respect to the relevant fields, modulo the bulk equations of motion. It follows that the equations constraining the fields on the boundary emerge.

With respect to the bosonic fields, setting the variation of the action $\delta S = 0$ gives us the following constraint on the boundary $\partial\mathbb{D}^2$:

$$\delta\bar{\mathcal{A}}(\partial_n\mathcal{A} + \varepsilon\mathtt{F}_{\varphi n}) + \delta\mathcal{A}(\partial_n\bar{\mathcal{A}} + \varepsilon\bar{\mathtt{F}}_{\varphi n} + \frac{\mathrm{i}}{\varepsilon}\partial_{\mathcal{A}}W) = 0, \tag{4.55}$$

provided the boundary condition that both A_n and X_φ vanish. The boundary conditions must be compatible with the equations of motion for the auxiliary fields $\mathtt{F}$ and $\bar{\mathtt{F}}$, as well as with the boundary Euler-Lagrange equation $\partial_{\mathcal{A}}W = 0$. Modulo them, (4.55) reduce to

$$\delta\bar{\mathcal{A}}(\partial_n\mathcal{A}) + \delta\mathcal{A}(\partial_n\bar{\mathcal{A}}) = 0. \tag{4.56}$$

We assume that the only constraint for the field variations $\delta(\mathcal{A}, \bar{\mathcal{A}})$ is that they must be tangent to γ:

$$(\delta\mathcal{A}, \delta\bar{\mathcal{A}}) \in T\gamma. \tag{4.57}$$

Given this, (4.56) then leads to

$$(\partial_n\mathcal{A}, \partial_n\bar{\mathcal{A}}) \in N\gamma, \tag{4.58}$$

where $T\gamma$ and $N\gamma$ denote the tangent and normal bundles of γ respectively.

Moreover, for X_m, there is another constraint arising via the following mechanism. As we have shown, the symplectic form in the space of complex gauge fields is given by (4.54). By assumption γ is gauge invariant, so the vector field

$$V_\epsilon = \mathrm{d}_A\epsilon + i[X, \epsilon] \tag{4.59}$$

generated by the infinitesimal gauge transformation with parameter $\epsilon \in \mathfrak{g}$ is tangent to γ. On the other hand, from the assumption that γ is a Lagrangian submanifold, ω vanishes restricted to γ. This means that the one-form

$$\iota_{V_\epsilon}\omega = \int_M \mathrm{Tr}(\mathrm{d}_A\epsilon \star \delta X - \delta A \wedge \star[X,\epsilon]) = -\int_M \mathrm{Tr}(\epsilon\delta(\mathrm{d}_A \star X)) \qquad (4.60)$$

annihilates any vector fields tangent to γ. By the boundary condition, variations of the complex gauge field are constrained to be tangent to γ, so $\delta(\mathrm{d}_A \star X) = 0$ on the boundary. It follows that $\star\mathrm{d}_A \star X \propto D^m X_m$ must be constant on the boundary. We will see that the only interesting value for this constant is zero; for the other values the path integral vanishes. Thus we impose

$$D^m X_m = 0 \qquad (4.61)$$

on the boundary. We will see that such a constraint will play an important role in the following one-loop computation.

With respect to the fermions, setting $\delta S = 0$ leads to the following constraint

$$\delta\rho_\varphi\bar{\mu}_{\varphi n} + \delta\bar{\eta}\rho_n = 0. \qquad (4.62)$$

In addition, the supersymmetry transformation laws give us

$$\delta_Q(\mathcal{A},\bar{\mathcal{A}}) = (\varepsilon\rho_\varphi, \bar{\eta}) \in T\gamma. \qquad (4.63)$$

$\varepsilon\rho_\varphi$ and $\bar{\eta}$ are the holomorphic and antiholomorphic components of $T\gamma$ respectively.

Since the variation of a vector in the tangent space should still belong to the tangent space, we should further obtain

$$\delta(\varepsilon\rho_\varphi, \bar{\eta}) \in T(\gamma). \qquad (4.64)$$

Mutiplying it by a ε factor, we can rewrite (4.62) as

$$(\varepsilon\delta\rho_\varphi, \delta\bar{\eta}) \begin{pmatrix} \bar{\mu}_{\varphi n} \\ \varepsilon\rho_n \end{pmatrix} = 0, \qquad (4.65)$$

which, together with (4.64), lead to

$$(\varepsilon\rho_n, \bar{\mu}_{\varphi n}) \in N(\gamma). \tag{4.66}$$

$\varepsilon\rho_n$ and $\bar{\mu}_{\varphi n}$ are the holomorphic and antiholomorphic components of $N\gamma$ respectively.

In addition, the boundary term $\mathcal{S}_{\text{boundary}}$ takes the Chern-Simons form, so it is invariant under large gauge transformations only if its Chern-Simons level is an integer. But the parameter ε is arbitrary, so for the path integral of the 5d theory to be well defined, we must remove the large gauge transformations for A on the boundary. Therefore, γ should be modulo all large gauge transformations of the gauge group $\mathcal{G}$.

After fixing our boundary conditions, let us proceed to the next step of doing the localization.

4.3.2.2 Saddle-point configurations

Gauge fixing and BRST symmetry

The path integral is evaluated by integrating out all the field configurations, modulo the gauge transformations. To eliminate all the degrees of gauge freedom, we introduce the Faddeev-Popov ghost field c to fix the gauge. We define the BRST transformations for the gauge-fixing as

$$\begin{aligned}
&\delta_B A_M = D_M c, \quad \delta_B c = \frac{\mathrm{i}}{2}\{c, c\}, \ \delta_B \bar{c} = \mathrm{i}B, \ \delta_B B = 0, \\
&\delta_B \Phi = \mathrm{i}[c, \Phi], \quad \text{with } \Phi = \{X_M, \mathtt{D}, \mathtt{F}\}, \\
&\delta_B \Psi = \mathrm{i}\{c, \Psi\}, \quad \text{with } \Psi = \{\rho, \bar{\eta}, \bar{\mu}, \lambda, \alpha, \zeta\},
\end{aligned} \tag{4.67}$$

where B is an auxiliary field. One can check that

$$\delta_B^2 = 0. \tag{4.68}$$

To remove the gauge redundancy, we add to our action the Q_B-exact term

$$\int \mathrm{d}^5x\sqrt{g}\,\frac{1}{2\xi}\,\mathrm{Tr}\,\delta_B\big(\bar{c}(g^{MN}\nabla_M A_N)-\frac{1}{4}\mathrm{i}\bar{c}B\big), \tag{4.69}$$

which becomes the gauge fixing term

$$\begin{aligned}\mathcal{S}^{5d}_{\mathrm{gf}} &= \int \mathrm{d}^5x\sqrt{g}\,\frac{1}{2\xi}\,\mathrm{Tr}\,\big(\bar{c}(\nabla_i D^i+\nabla_m D^m)c+(\nabla_i A^i+\nabla_m A^m)^2\big)\\ &= \int \mathrm{d}^5x\sqrt{g}\,\frac{1}{2\xi}\,\mathrm{Tr}\,\big(\bar{c}\nabla_i D^i c+\bar{c}\nabla_m D^m c+(\nabla_i A^i)^2+(\nabla_m A^m)^2+2(\nabla_i A^i)(\nabla_m A^m)\big)\end{aligned} \tag{4.70}$$

after integrating out the auxiliary filed B, where $1/2\xi$ is an arbitrary constant.

Given that the theory being gauge-fixed, we move on to find out our saddle-point configurations.

Saddle-point configurations

To fix the saddle-point configurations, we first resort to the Q-exactness of $\mathcal{S}^{5d}_{\mathrm{V}}$ and $\mathcal{S}^{5d}_{\mathrm{C}}$. We can do the following rescalings:

$$\mathcal{S}^{5d}_{\mathrm{V}} \to t\mathcal{S}^{5d}_{\mathrm{V}}, \quad \mathcal{S}^{5d}_{\mathrm{C}} \to t\mathcal{S}^{5d}_{\mathrm{C}}, \tag{4.71}$$

with t a constant.

Under the rescaling, the action is rescaled by

$$t\mathcal{S}^{5d}_{\mathrm{V}}+t\mathcal{S}^{5d}_{\mathrm{C}}+\mathcal{S}^{5d}_{\mathrm{gf}}+\mathcal{S}^{5d}_{\mathrm{W}}+\mathcal{S}^{5d}_{\mathrm{boundary}}, \tag{4.72}$$

where note that the total derivative terms in $\mathcal{S}_{\mathrm{V}}$ and $\mathcal{S}_{\mathrm{C}}$ vanish, due to the boundary condition $X_n=0$.

In addition, we notice that there is a Q-exact term $\mathrm{Tr}\,\delta(\mathtt{F}\wedge\star\bar{\mu})$ in the action. Due to the Q-exactness, without changing the theory we can further rescale it by

$$t\,\mathrm{Tr}\,\delta(\mathtt{F}\wedge\star\bar{\mu}) \to st\,\mathrm{Tr}\,\delta(\mathtt{F}\wedge\star\bar{\mu}), \tag{4.73}$$

with s a constant. The purpose of this rescaling will be clear immediately.

After doing the above rescalings, we are now ready to fix the saddle-point configurations. Firstly, let us integrate out all the auxiliary fields. After integrating out the auxiliary field $\mathtt{F}$, the terms that originally contain $\mathtt{F}$ carry the factor $1/s$. Therefore these terms vanish when we take

$$s \to \infty. \tag{4.74}$$

And after integrating out the auxiliary field $\mathtt{D}$, we obtain the term $t(D_m X^m)^2$.

Secondly, we take $t \to +\infty$. The path-integral then localizes to the configurations which set the bosonic terms in $\mathcal{S}_{\mathrm{V}}^{5d}$ and $\mathcal{S}_{\mathrm{C}}^{5d}$. Thus finally we obtain the following saddle-point configurations

$$F_{ij} = D_i X_j = D_m X_i = D_i X_m = F_{im} = D_m X^m = [X_i, X_j] = \mathcal{R} X_i X^i = [X_i, X_m] = 0 \tag{4.75}$$

As $\mathbb{D}^2$ is simply connected, by $F_{ij} = 0$ all the saddle-point configurations of A_i are gauge equivalent to $A_i = 0$. Since after gauge-fixing there is no gauge freedom in our theory, we just choose $A_i = 0$ to be the saddle-point configuration for A_i. Since the theory is quasi-topological on $\mathbb{D}^2$, we can deform the disk such that $\mathcal{R}$ is non-vanishing (as we shall see, $\mathbb{D}^2$ will be deformed to S^2 for localization computation), which then sets $X_i = 0$ as the saddle-point configuration. In this context, (4.75) reduces to

$$\begin{aligned} A_i = \partial_i A_m = \partial_i X_m &= 0, \\ D_m X^m = \frac{\mathrm{i}}{2} D_m (\bar{\mathcal{A}} - \mathcal{A})^m &= 0, \end{aligned} \tag{4.76}$$

by which we can see that the choice of the boundary condition (4.61) is necessary.

Here note that the saddle-point configurations of $\mathcal{A} = A_m + \mathrm{i} X_m$ are constants on $\mathbb{D}^2$ but can fluctuate on M, and we denote them by $\mathcal{A}_0$. By the constraint $D_m X^m = 0$, for $\mathcal{A}_{m0} = A_{m0} + \mathrm{i} X_{m0}$, the real part A_{m0} and the imaginary part X_{m0} are both fixed to their gauge inequivalent configurations in the path integral (where A_{m0} is *a priori* gauge-fixed in the path integral, as it is the gauge field in the 5d theory). Provided this, we can treat $\mathcal{A}_0$ as the complex gauge connection of the complex gauge group $\mathcal{G}_{\mathbb{C}}$ (which is the complexification of the gauge group $\mathcal{G}$ of the 5d theory), where $D_m X^m = 0$ serves to fix the gauge transformations for the imaginary part of the complex gauge field. In the path integral the complex gauge fields $\mathcal{A}_0$ take values in the Lie algebra of the complex group $\mathcal{G}_{\mathbb{C}}$, modulo the complex gauge transformations [3].

Finally, let us try to foresee what we will obtain via localization. Note that at the saddle-point configurations only the term $\mathcal{S}^{5d}_{\text{boundary}}$ in the action survives with the other terms all vanishing. Thus at the limit $t \to +\infty$, the action reduces to

$$\mathcal{S}^{5d} = t\tilde{\mathcal{S}}^{5d}_{\text{V}} + t\tilde{\mathcal{S}}^{5d}_{\text{C}} + \mathcal{S}^{5d}_{\text{gf}} + \mathcal{S}^{5d}_{\text{boundary}}(\mathcal{A}_0), \tag{4.77}$$

where $\tilde{\mathcal{S}}^{5d}_{\text{V}}$ and $\tilde{\mathcal{S}}^{5d}_{\text{C}}$ are expanded around the saddle-point configurations, while $\mathcal{S}^{5d}_{\text{boundary}}(\mathcal{A}_0)$ depends only on $\mathcal{A}_0$. $\mathcal{S}^{5d}_{\text{gf}}$ just plays the role of gauge-fixing the theory. Therefore, by integrating out all fluctuations of the fields, the partition function of the 5d theory reduces to

$$\int D\mathcal{A}_0 Z_1(\mathcal{A}_0)\text{exp}\left(-\frac{2\text{i}\pi}{\varepsilon}(W(\mathcal{A}_0) + W_0)\right), \tag{4.78}$$

in which $Z_1(\mathcal{A}_0)$ represents the one-loop determinant factor, given the condition that the zero modes of fermions are absent (such a condition will be acquired in the succedent section). This is the partition function of the holomorphic part of the complex Chern-Simons theory with complex gauge group $\mathcal{G}_{\mathbb{C}}$, where

$$W(\mathcal{A}_0) = \frac{1}{2}\text{Tr}\int_M \text{CS}(\mathcal{A}_0) = \frac{1}{2}\int_M \text{Tr}(\mathcal{A}_0 \wedge \text{d}\mathcal{A}_0 - \frac{2\text{i}}{3}\mathcal{A}_0 \wedge \mathcal{A}_0 \wedge \mathcal{A}_0). \tag{4.79}$$

Next, let us evaluate the one-loop determinant $Z_1(\mathcal{A}_0)$, and we will find that it is independent of the saddle-point configurations $\mathcal{A}_0$ and thus can be absorbed into the measure $\int D\mathcal{A}_0$.

4.3.2.3 One-loop determinants

To evaluate the one-loop determinants, let us write down the quadratic terms of the action expanded around the saddle-point configurations. Note that in the following discussion we assume that the fermion zero modes are absent, a proof of which will be given shortly.

[3] A more rigorous discussion can be found in [40].

First, we expand the ghost by

$$c = c_0 + \tilde{c}, \tag{4.80}$$

where c_0 denote the modes which are constant on $\mathbb{D}^2$ but can still fluctuate on M. Then for the gauge-fixing term $\mathcal{S}^{5d}_{\text{gf}}$, it can be written as

$$\begin{aligned}\mathcal{S}^{5d}_{\text{gf}} = \int \mathrm{d}^5x\sqrt{g}\,\frac{1}{2\xi}\,\mathrm{Tr}\big(&\bar{c}_0\nabla_m D^m c_0 + (\nabla_m A^{m0})^2 \\ &+ \tilde{\bar{c}}\nabla_i\nabla^i\tilde{c} + \tilde{\bar{c}}\nabla_m D^m\tilde{c} \\ &+ (\nabla_i\tilde{A}^i)^2 + (\nabla_m\tilde{A}^m)^2 + 2(\nabla_i\tilde{A}^i)(\nabla_m\tilde{A}^m).\end{aligned} \tag{4.81}$$

Note that the cross terms consisting of both the constant and nonconstant modes on $\mathbb{D}^2$ simply vanish due to the orthogonality of the eigenmodes. As $1/2\xi$ is arbitrary, let us take $1/2\xi = t$ in the following computation.

For simplifying the analysis, we rescale the *nonzero modes* of the fields by

$$\begin{aligned}&t^{-1/2}\tilde{A}_i, \quad t^{-1/2}\tilde{X}_i, \quad \frac{t^{-1/2}}{s}\alpha, \quad \frac{t^{-1/2}}{s}\zeta, \quad st^{-1/2}\lambda, \\ &t^{-1/2}\tilde{\mathcal{A}}_m, \quad \frac{t^{-1/2}}{s}\rho, \quad \frac{t^{-1/2}}{s}\bar{\mu}, \quad st^{-1/2}\bar{\eta}, \\ &t^{-1/2}\tilde{c}, \quad t^{-1/2}\tilde{\bar{c}}, \quad t^{-1/2}c_0, \quad t^{-1/2}\bar{c}_0.\end{aligned} \tag{4.82}$$

Then, in the limit of $\kappa, s \to +\infty$, in the action the fluctuating terms expanded around the saddle-point configurations reduce to:

$$t\tilde{\mathcal{S}}^{5d}_{\text{V}} + t\tilde{\mathcal{S}}^{5d}_{\text{C}} + \tilde{\mathcal{S}}^{5d}_{\text{gf}} = \tilde{\mathcal{S}}^{5d}_{b} + \tilde{\mathcal{S}}^{5d}_{f} + \mathcal{S}_{\text{gf0}}, \tag{4.83}$$

where

$$\begin{aligned}\tilde{\mathcal{S}}^{5d}_{b} = \int \mathrm{d}^5x\sqrt{g}\,\mathrm{Tr}\Big(&\frac{1}{2}(\nabla_i\tilde{A}_j - \nabla_j\tilde{A}_i)(\nabla^i\tilde{A}^j - \nabla^j\tilde{A}^i) + \nabla_i\tilde{X}_j\nabla^i\tilde{X}^j + \nabla_i\tilde{\mathcal{A}}_m\nabla^i\tilde{\bar{\mathcal{A}}}^n + \mathcal{D}_m\tilde{A}_i\bar{\mathcal{D}}^m\tilde{A}^i \\ &+ \mathcal{D}_m\tilde{X}_i\bar{\mathcal{D}}^m\tilde{X}^i + (D_m\tilde{X}^m - \mathrm{i}\tilde{A}_m X_0^m)^2 + (\nabla_i\tilde{A}^i)^2 + (\nabla_m\tilde{A}^m)^2 \\ &+ 2\mathrm{i}\nabla_i\tilde{X}_m X^{0m}\tilde{A}^i\Big),\end{aligned} \tag{4.84}$$

$$
\begin{aligned}
\tilde{\mathcal{S}}_f^{5d} = \int \mathrm{d}^5x\sqrt{g}\,\mathrm{Tr}\big(& -2\alpha\nabla_i\lambda^i - 2\epsilon^{ij}\zeta\nabla_i\lambda_j \\
& -\rho_m^i\nabla_i\bar{\eta}^n + \epsilon^{ij}\nabla_i\rho_{jm}\bar{\mu}^m \\
& +2\mathrm{i}\rho_{im}\bar{\mathcal{D}}^m\lambda^i + \mathrm{i}\alpha_i^i\mathcal{D}_m\bar{\eta}^m + \mathrm{i}\zeta\mathcal{D}_m\bar{\mu}^m \\
& +\tilde{c}\nabla_i\nabla^i\tilde{\bar{c}} + \tilde{\bar{c}}\nabla_m D^m\tilde{c}\big)
\end{aligned}
\tag{4.85}
$$

and the last term

$$
\mathcal{S}_{\mathrm{gf0}} = \int \mathrm{d}^5x\sqrt{g}\;\mathrm{Tr}\left(\bar{c}_0\nabla_m D^m c_0 + t(\nabla_m A^{m0})^2\right), \tag{4.86}
$$

which serves to gauge-fix the saddle-point configurations of A_m. Here the covariant derivatives D_i and $\mathcal{D}_m$ only depend on the saddle-point configurations of the bosonic fields. And since the saddle-point configurations of A_i are zero, D_i reduce to ∇_i.

In order to evaluate the one-loop determinants, one need to write down the Laplacian matrices by performing integration by parts for (4.84) and (4.85). However, due to the existence of the boundary of $\mathbb{D}^2$, some boundary terms can arise when we perform integration by parts. Luckily, this complexity can be resolved in the following way.

According to the topological property of the action, we have the freedom to deform the metric of $\mathbb{D}^2$ in the bulk theory; in addition, our boundary condition is independent of the metric component h_{nn}. Therefore, we can deform $\mathbb{D}^2$ to a sphere S^2, with a small disk removed. In spherical coordinates the metric takes the form $R^2(\mathrm{d}\vartheta^2 + \sin^2\vartheta\mathrm{d}\varphi^2)$, with $0 \le \vartheta \le \vartheta|_{\partial\mathbb{D}^2}$. Provided the boundary condition remains the same, we can vary R but keep $h_{\varphi\varphi}|_{\partial\mathbb{D}^2} = R^2\sin^2\vartheta_{\partial\mathbb{D}^2}$ fixed. When $R\to\infty$, $\sin\vartheta_{\partial\mathbb{D}^2}\to 0$, so $\mathbb{D}^2$ is deformed to a sphere with a tiny hole at the pole $\vartheta = \pi$. In such a limit, we can deform the worldsheet to become the whole S^2, by mapping the boundary state to some Q-invariant operator inserted at $\vartheta = \pi$ (which selects the nonvanishing modes just as the boundary condition does). Thus, there would be no boundary terms arising.

By the fact that the theory is invariant under the rescaling the size of the S^2 deformed from $\mathbb{D}^2$, we can shrink the size of S^2 to be very small. Since the eigenvalue of the non-constant mode on S^2 is proportional to $1/r$ (with r the radius), in the limit $r \to 0$, the contributions of the other terms in (4.84) and (4.85) to the one-loop determinants all decouple. Thus the contributions to the one-loop determinants only come from the contributions of the quadratic terms with the Laplacian of S^2:

$$\begin{aligned}\int \mathrm{d}^5x\sqrt{g}\mathrm{Tr}\big(\tilde{A}_i\nabla_k\nabla^k\tilde{A}^i &+ \tilde{X}_j\nabla_i\nabla^i\tilde{X}^j + \tilde{\mathcal{A}}_m\nabla_i\nabla^i\tilde{\bar{\mathcal{A}}}^n \\ &- 2\alpha\nabla_i\lambda^i - 2\epsilon^{ij}\zeta\nabla_i\lambda_j \\ &- \rho^i_m\nabla_i\bar{\eta}^n + \epsilon^{ij}\nabla_i\rho_{jm}\bar{\mu}^m \\ &+ \tilde{c}\nabla_i\nabla^i\tilde{\bar{c}}\big).\end{aligned} \tag{4.87}$$

Therefore, in the absence of the constant modes on S^2, the one-loop determinants are independent of the saddle-point configurations $\mathcal{A}_0$. We will soon show that the absence of constant modes on S^2 is indeed true.

Thus finally, and most importantly, as the one-loop determinant factor is independent of the saddle-point configurations of $\mathcal{A}_m$, it can be absorbed into the measure $D\mathcal{A}_0$, which leads to (4.78) becoming [4]

$$\int D\mathcal{A}_0 \exp\left(-\frac{2\mathrm{i}\pi}{\varepsilon}(W(\mathcal{A}_0)+W_0)\right), \tag{4.88}$$

where locally W_0 is a constant independent of $\mathcal{A}_0$, and

$$W(\mathcal{A}_0) = \frac{1}{2}\mathrm{Tr}\int_M \mathrm{CS}(\mathcal{A}_0) = \frac{1}{2}\int_M \mathrm{Tr}(\mathcal{A}_0\wedge \mathrm{d}\mathcal{A}_0 - \frac{2\mathrm{i}}{3}\mathcal{A}_0\wedge\mathcal{A}_0\wedge\mathcal{A}_0). \tag{4.89}$$

Thus the 3d complex Chern-Simons theory is successfully obtained by localization.

Absence of zero modes on S^2

We just assumed that there are no constant modes on S^2 in evaluating the one-loop determinants. In the following we will prove that this assumption is indeed true.

[4]Here it is easy to see that deforming $\mathbb{D}^2$ to S^2 does not alter the saddle-point configurations we obtained in the previous section.

For the bosonic fields $\tilde{A}_i$ and $\tilde{X}_i$, it is easy to see that they both contain no constant modes on S^2. For A_m, by definition, all the constant modes are the saddle-point configurations, thus $\tilde{A}_m$ contain no constant modes. For X_m, the saddle-point configurations are defined by

$$\partial_i X_m,\ D_m X^m = 0. \tag{4.90}$$

Consequently, there are constant modes of X_m on S^2 which do not satisfy $D_m X^m$ contained in $\tilde{X}_m$. However, the boundary condition (4.61) kills all these modes. (Note that on the S^2 the boundary condition is imposed via the Q-invariant operator inserted at $\vartheta = \pi$.) Therefore, we conclude that there are no constant modes for $\tilde{X}_m$ on S^2.

For the fermionic fields λ, α and ζ, the fermion quadratic terms can be written as

$$(\star\alpha_{z\bar{z}} - \star\zeta_{z\bar{z}},\quad \star\alpha_{z\bar{z}} + \star\zeta_{z\bar{z}}\ ) \begin{pmatrix} \partial_z & 0 \\ 0 & \partial_{\bar{z}} \end{pmatrix} \begin{pmatrix} \lambda_{\bar{z}} \\ \lambda_z \end{pmatrix}. \tag{4.91}$$

On S^2 there are no harmonic one-forms, thus there are no zero modes for λ. Furthermore, harmonic two-forms are Hodge duals of constants, and neither having $\star\alpha$ constant nor $\star\zeta$ constant is compatible with the boundary condition $\partial_n \alpha_{n\varphi} = \zeta_{n\varphi} = 0$. (If we describe our worldsheet as the Riemann sphere parametrized by $z = ne^{i\varphi}$, then taking Fubini-Study metric, we can have the zero mode of α behaving as $\alpha_{n\varphi} \sim n/(1+n^2)^2$ near the boundary $n = 0$.)

Similarly, for the other fermion fields, the quadratic terms can be written as

$$(-\bar{\mu}_{z\bar{z}} - \bar{\eta},\quad \bar{\mu}_{z\bar{z}} - \bar{\eta}\ ) \begin{pmatrix} \partial_z & 0 \\ 0 & \partial_{\bar{z}} \end{pmatrix} \begin{pmatrix} \rho_{\bar{z}} \\ \rho_z \end{pmatrix}. \tag{4.92}$$

Then using the same argument we used for λ, we conclude that there are no zero modes for ρ. Here we can see that the zero modes for $\bar{\mu}_{z\bar{z}}$ and η are constants; but to further prove that there are actually no zero modes for them, we have to take account of the constraints imposed by the boundary condition, which is given by (4.63) and (4.66). As the path integral is invariant under rescaling the size of the S^2, we take the limit whereby the size is very small. In this limit, the non-zero modes are very massive and decouple. Thus the boundary condition can be imposed on the zero modes independently. So (4.63) and (4.66) reduce to

$$(0, \bar{\eta}_0) \in T\gamma, \quad (0, \bar{\mu}_0) \in N\gamma. \tag{4.93}$$

As γ is a Lagrangian submanifold of a Kähler manifold, its complex structure J maps the tangent space $T_p\gamma$ at an arbitrary point p isometrically onto the corresponding normal space $N_p\gamma$: $J(T_p\gamma) = N_p\gamma$. It then follows that $J(0, \bar{\eta}_0) = (0, \mathrm{i}\bar{\eta}_0) \in N\gamma$, which on the boundary leads to $\bar{\eta}_0 = 0$, since $T\gamma \cap N\gamma = 0$; likewise, on the boundary $\bar{\mu}_0 = 0$. Provided this, we thus conclude that the boundary condition kills all the zero modes for $\bar{\eta}$ and $\bar{\mu}$.

4.4 Conclusion

To conclude our discussion, in the final section we interpret the results we obtained about the Ω-deformed twisted 5d MSYM theory from the point of view of the 3d-3d correspondence. This allows us to establish the correspondence between the 3d $\mathcal{N} = 2$ superconformal theory $T[M]$ and analytically continued Chern–Simons theory on M. Furthermore, we will see that our construction of the 5d theory, together with the 3d-3d correspondence, implies a mirror symmetry between Ω-deformed 2d theories.

4.4.1 $T[M]$ and analytically continued Chern–Simons theory

Consider the $(2,0)$ theory on $S^1 \times_V \Sigma \times M$, with S^1 a circle of radius R and V a Killing vector field on Σ. Here, the space $S^1 \times_V \Sigma$ is a nontrivial Σ-fibration over S^1, constructed from the trivial fibration $[0, 2\pi R] \times \Sigma$, by gluing the two ends of the interval $[0, 2\pi R]$ with an action of the isometry $\exp(2\pi RV)$ on the fiber Σ. The structure group of the spinor bundle of this space is reduced to $\mathrm{Spin}(2)_\Sigma \times \mathrm{Spin}(3)_M$, and the R-symmetry group of the theory is $\mathrm{Spin}(5)_R$. This is just like the case of 5d MSYM theory on $\Sigma \times M$. Thus, we can consider topological twisting analogous to the one applied to that theory.

It is well known that for flat spacetime, the $(2,0)$ theory compactified on S^1 is equivalent, at low energies, to 5d MSYM theory with gauge coupling $e^2 =$

$4\pi^2 R$. In view of this relation, we propose that at energies much smaller than $1/R$, the above twisted $(2,0)$ theory on $S^1 \times_V \Sigma \times M$ is equivalent to the Ω-deformed twisted 5d MSYM theory on $\Sigma \times M$ constructed in the previous section, with the same gauge coupling and the Ω-deformation given by a Killing vector field proportional to V.

Another regime that is relevant to us is the one in which energies are much smaller than $1/L$, where L is the length scale of M. In this regime, the $(2,0)$ theory compactified on M gives $T[M]$ by definition. Hence, the twisted $(2,0)$ theory reduces to a topologically twisted version of $T[M]$ on $S^1 \times_V \Sigma$.

Based on our proposal and this observation, we can show that the Ω-deformed twisted 5d MSYM theory is equivalent to the twisted $T[M]$. The argument goes as follows.

We fix an energy scale E, and consider the twisted $(2,0)$ theory on $S^1 \times_V \Sigma \times M$ with R, $L \ll 1/E$. This theory can be described either as the Ω-deformed twisted 5d MSYM theory on $\Sigma \times M$, with e^2 and M small, or as the twisted $T[M]$ on $S^1 \times_V \Sigma$, with the S^1 small. The 5d theory is topological on M, so we can scale up M if we wish. Likewise, the 3d theory is independent of R and we can set it to any value as long as we keep unchanged the isometry $\exp(2\pi RV)$ (and other possible fugacity parameters associated to boundaries in M), for correlation functions on $S^1 \times_V \Sigma$ are supersymmetric indices. (See e.g. [43] for more discussions on this point.)

The last statement suggets that the 5d theory depends on e^2 only through the combination $e^2 V$, and this is indeed true. To see this, we consider a Q-exact deformation of the action similar to the one used in the derivation of the localization formula for $\Sigma = \mathbb{D}^2$ in section 4.3.2. After such a Q-exact deformation, only S_V, S_C and the boundary term in S_W are relevant for the computation of the path integral. The claim then follows from the fact that the dependence on e^2 coming from the first two is Q-exact, while the boundary term of the action depends on e^2 through the factor $1/e^2\varepsilon$. Thus, we can rescale e^2 to any value, if we simultaneously rescale V to keep $e^2 V$ fixed.

Since the 5d and 3d theories are different descriptions of the same 6d theory, they are equivalent, and this is valid at any energy scale E, for any values of e^2 and R, and for any metric on M. Therefore, we conclude that the Ω-deformed twisted 5d MSYM theory on $\Sigma \times M$ is equivalent to the twisted $T[M]$ on $S^1 \times_V \Sigma$. Our argument is depicted in fig. 4.1.

$$
\begin{tikzcd}
& \boxed{(2,0) \text{ theory on } S^1 \times_V \Sigma \times M} \arrow[dl] \arrow[dr] & \\
\boxed{\Omega\text{-def'd 5d MSYM on } \Sigma \times M} \arrow[rr, leftrightarrow] & & \boxed{T[M] \text{ on } S^1 \times_V \Sigma}
\end{tikzcd}
$$

FIGURE 4.1: Equivalence between the Ω-deformed twisted 5d MSYM theory and the twisted $T[M]$

Now we take $\Sigma = \mathbb{D}^2$. In this case we have shown that the Ω-deformed twisted 5d MSYM theory is equivalent to analytically continued Chern–Simons theory. Combined with the equivalence just discussed, this establishes the correspondence between $T[M]$ and the latter theory (fig. 4.2).

$$
\begin{tikzcd}
& \boxed{\Omega\text{-def'd 5d MSYM on } \mathbb{D}^2 \times M} \arrow[dl] \arrow[dr, leftrightarrow] & \\
\boxed{\text{analytically cont'd CS on } M} \arrow[rr, leftrightarrow] & & \boxed{T[M] \text{ on } S^1 \times_V \mathbb{D}^2}
\end{tikzcd}
$$

FIGURE 4.2: Correspondence between $T[M]$ and analytically continued Chern–Simons theory

Let us briefly comment on an alternative explanation for this correspondence, proposed by Beem et al. [43]. Their approach starts with the same 6d setup as ours, namely the $(2,0)$ theory on $S^1 \times_V \mathbb{D}^2 \times M$. The main difference is that in their case, in addition to reduction on the S^1, one considers deforming $\mathbb{D}^2$ to a cigar shape and reducing the theory on the circle fibers of $\mathbb{D}^2$. After doing so, one has a twisted $\mathcal{N} = 4$ super Yang–Mills theory on the product of an interval and M. Then one can invoke an argument given in [84, 85] and show that the system is equivalent to the Chern–Simons theory. Our derivation has the advantage that it avoids questions concerning the singular point of the geometry, that is the tip of the cigar, where the circle fiber shrinks to a point and the analysis becomes difficult.

In deriving the correspondence between $T[M]$ and analytically continued Chern–Simons theory, we set $\Sigma = \mathbb{D}^2$ and impose boundary conditions of a specific type. Similar localization computations may be carried out for other choices of Σ and boundary conditions, and may lead to yet unknown correspondences.

4.4.2 Ω-deformed mirror symmetry

The equivalence between the Ω-deformed twisted 5d MSYM theory and the twisted $T[M]$ implies more than just the correspondence discussed above. We can use it to find another interesting correspondence which relates two Ω-deformed 2d theories.

Consider 5d MSYM theory, compactified and topologically twisted on M. In the limit where M is very small, it becomes an $\mathcal{N} = (2,2)$ theory $\widetilde{T}[M]$ in two dimensions. An analysis along the lines of [79] shows that $\widetilde{T}[M]$ is a Landau–Ginzburg model whose target space is the moduli space $\mathcal{M}_{\text{flat}}$ of complex flat connections on M, assuming that the flat connections are irreducible.[5]

If we instead start from the Ω-deformed twisted 5d MSYM theory on $\Sigma \times M$, then we obtain an Ω-deformed, twisted version of $\widetilde{T}[M]$ on Σ. The model is more precisely B-twisted, as our construction of the 5d theory is based on a B-twisted gauge theory, and the chiral multiplets of the model simply come from their counterparts in the 5d theory, containing $\mathcal{A}_m$. Alternatively, one may note that generically $U(1)_V$ would be broken by the superpotential, so the twisting should be done with $U(1)_A$. (If the model happens to have a quasi-homogeneous superpotential, one can deform the 5d theory so that nonhomogenous terms are generated; then one knows that the 2d theory is B-twisted, as the twisting does not change under such a deformation.)

On the other hand, $T[M]$ compactified on S^1 reduces to an $\mathcal{N} = (2,2)$ theory $\widehat{T}[M]$ in the limit $R \to 0$. So if we instead start with the twisted version of $T[M]$ formulated on $S^1 \times_V \Sigma$, then we get an Ω-deformed twisted $\widehat{T}[M]$ on Σ.

[5] In general, the Landau–Ginzburg model description breaks down at reducible flat connections due to appearance of extra massless modes on M coming from A_μ, σ_μ and their superpartners. This echoes the observation made in [34, 82] that the construction of $T[M]$ proposed in [33, 83] really captures only the subsector of the full theory, obtained by truncation to the irreducible connections.

Now, combining the facts that (1) the Ω-deformed twisted 5d MYSM theory is topological on M; (2) the twisted $T[M]$ on $S^1 \times_V \Sigma$ is independent of R (as long as RV and other fugacities are fixed); and (3) these two theories are equivalent, we deduce that the Ω-deformed twisted $\widetilde{T}[M]$ is equivalent to the Ω-deformed twisted $\widehat{T}[M]$ (fig. 4.3).

$$\begin{tikzcd}
\boxed{\Omega\text{-def'd 5d MSYM on } \Sigma \times M} \arrow[r, leftrightarrow] \arrow[d] & \boxed{T[M] \text{ on } S^1 \times_V \Sigma} \arrow[d] \\
\boxed{\Omega\text{-def'd } \widetilde{T}[M] \text{ on } \Sigma} \arrow[r, leftrightarrow] & \boxed{\Omega\text{-def'd } \widehat{T}[M] \text{ on } \Sigma}
\end{tikzcd}$$

FIGURE 4.3: Ω-deformed mirror symmetry

This equivalence may be thought of as a mirror symmetry. The reason is that while the twisted 5d MSYM theory reduced on M gives rise to a B-twisted Landau–Ginzburg model, reduction of the twisted $T[M]$ on the S^1 produces an *A-twisted* gauge theory, if $T[M]$ is realized as gauge theory as in [33, 68]; in particular, it can flow to an A-twisted sigma model in the infrared. This may be seen from the fact that a scalar in the vector multiplet of the 2d theory comes from a component of the 3d gauge field, which is neutral under the R-symmetry $U(1)_R$ used in the topological twist of the 3d theory. Since the scalar is charged under the axial R-symmetry $U(1)_A$, it follows that $U(1)_R$ becomes the vector R-symmetry $U(1)_V$.

Specializing to the case $\Sigma = \mathbb{D}^2$, we can place the correspondence between $T[M]$ and analytically continued Chern–Simons theory (fig. 4.2) and the one between $\widetilde{T}[M]$ and $\widehat{T}[M]$ (fig. 4.3) in a single diagram (fig. 4.4). The result is an intriguing triangle of correspondences that connects analytically continued Chern–Simons theory, $\widetilde{T}[M]$ and $\widehat{T}[M]$.

$$\begin{tikzcd}
& \boxed{\text{analytically cont'd CS on } M} \arrow[dl, leftrightarrow] \arrow[dr, leftrightarrow] & \\
\boxed{\Omega\text{-def'd B-tw'd } \widetilde{T}[M] \text{ on } \mathbb{D}^2} \arrow[rr, leftrightarrow] & & \boxed{\Omega\text{-def'd A-tw'd } \widehat{T}[M] \text{ on } \mathbb{D}^2}
\end{tikzcd}$$

FIGURE 4.4: A triangle of correspondences

Chapter 5

Summary and Outlook

Let us summarize the key results revealed in this thesis. We will then conclude with possible directions suggested by our results for future research.

First, some of the topologically twisted supersymmetric gauge theories are topological quantum field theories (while others are just partially topological). Topological quantum field theories open a door for studying mathematics via physics, giving rise to rich results in topology. The case we studied in chapter 2 is a successful example. We constructed a topological Chern-Simons sigma model on a Riemannian three-manifold M with gauge group $\mathcal{G}$ whose hyperkähler target space X is equipped with a $\mathcal{G}$-action satisfying the condition (2.3). Via a perturbative computation of its partition function, we obtained new topological invariants of M that define new weight systems which are characterized by both Lie algebra structure and hyperkähler geometry. In canonically quantizing the sigma model, we found that the partition function on certain M can be expressed in terms of Chern-Simons knot invariants of M and the intersection number of certain $\mathcal{G}$-equivariant cycles in the moduli space of $\mathcal{G}$-covariant maps from M to X. We also constructed supersymmetric Wilson loop operators, and via a perturbative computation of their expectation value, we obtained new knot invariants of M that define new knot weight systems which are also characterized by both Lie algebra structure and hyperkähler geometry.

Second, as topologically twisted supersymmetric gauge theories correspond to certain topological BPS sectors of untwisted theories, they are good candidates for studying low energy physics. As revealed by our results in chapter 3, in the low energy limit $\mathcal{N} = 2$ supersymmetric gauge theories can be described by quantum integrable systems. Specifically in our context, we studied the $\mathcal{N} = 2$ supersymmetric gauge theory on $S^2 \times S^1 \times \mathbb{R}$, which is topologically twisted along $S^1 \times \mathbb{R}$. Via compactification and dualization, its low-energy effective theory in four dimensions can reduce to a sigma model on $S^2 \times \mathbb{R}$, which we constructed by lifting. By localization on the S^2, we reduced this sigma model to a quantum integrable system, with the Planck constant set by the inverse of the radius of the S^2. Thus, we showed that the low-energy dynamics of a BPS sector of the $\mathcal{N} = 2$ supersymmetric gauge theory is described by a quantum integrable system. If the sphere is replaced with a hemisphere, then our system reduces to an integrable system of the type studied by Nekrasov and Shatashvili, whereby we established a correspondence between the effective prepotential of the gauge theory and the Yang-Yang function of the integrable system.

Third, topologically twisted supersymmetric gauge theories can be applied in studying various correspondences. Since (with respect to topologically twisted supercharges) certain BPS sectors are protected against the localization and compactification procedures, starting from a topologically twisted theory on $X \times M$ (where X and M are d_X and d_M-dimensional manifolds respectively,) the d_X/d_M correspondence can be established by identifying the Q-invariant quantities of the two theories $T[M]$ and $T[X]$ obtained respectively by compactification on M and localization on X. In chapter 4, we formulated a five-dimensional super-Yang-Mills theory (SYM) on $\mathbb{D}^2_\varepsilon \times M$, where the 5d SYM is topologically twisted along the three-manifold M, and its supercharge is the Ω-deformation of the B-twisted $\mathcal{N} = (2,2)$ supercharges on the disk $\mathbb{D}^2$. Our 5d SYM can be viewed as the compactification of the 6d $(2,0)$ superconformal field theory on S^1. By localization on $\mathbb{D}^2$, our 5d SYM reduced to the holomorphic part of the complex Chern-Simons theory. As a consequence, our result also indicated the existence of a mirror symmetry in two-dimensional Ω-deformed gauge theories.

Thus, we have seen that the topologically twisted supersymmetric theories give rise to various interesting and useful results in both physics and mathematics. Clearly, the study in the three cases can be deepened and expanded. For the first case, novel and sophisticated mathematical methods could be invented to explicitly verify the cancellation of the metric-variation of the partition function, giving our result a more rigorous proof. For the second case, one can derive the sigma model starting from the four dimensional effective theory by dimensional reduction and dualization, making our work more complete. For the last case, one can further obtain the $\mathcal{N} = 2$ superconformal theory $T[M]$ (which is the compactification of the corresponding 3d SCFT on S^1) by compactifying the 5d SYM on M, which could lead to a mirror symmetry being revealed. Besides these possible extensions, importantly, other topological supersymmetric gauge theories in different dimensions with different supersymmetries can be formulated and applied into different physical or mathematical contexts, and using similar methods applied in this thesis, we look forward to finding other intriguing and inspiring results in quantum field theory, string theory and topology. The possibilities are wide open and exciting.

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